M1220-2 **Midterm 2** **Spring 2005**

Quiz Scores (out of 100): n = 139; mean = 65

25th percentile =541; median (50th percentile) = 70; 75th percentile = 79

1.(40 pts) **Evaluate each of the following integrals. Give a numerical answer for the definite integral.**

i.

$$\int \frac{\cosh(\sqrt{x})}{\sqrt{x}} dx$$

Letting $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}$ we have

$$\int \frac{\cosh(\sqrt{x})}{\sqrt{x}} dx = 2 \int \cosh(u) du = 2\text{sinh}(u) + C = 2\text{sinh}(\sqrt{x}) + C$$

.

ii.

$$\int \ln(4x^3) dx$$

Using integration by parts and letting $u = \ln(4x^3), du = (3/x)dx$, and $dv = dx, v = x$, we have

$$\int \ln(4x^3) dx = x\ln(4x^3) - \int 3dx = x\ln(4x^3) - 3x + C$$

iii. (Hint: Remember to divide first.)

$$\int \frac{x^3 + x^2 + 1}{x(x+1)} dx$$

Dividing $x(x+1) = x^2 + x$ into $x^3 + x^2 + 1$ we have, $\frac{x^3+x^2+1}{x(x+1)} = x + \frac{1}{x(x+1)}$. Now setting

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{(A+B)x + A}{x(x+1)}$$

we have the equations, $A + B = 0, A = 1$. Solving we get $A = 1, B = -1$. So,

$$\int \frac{x^3 + x^2 + 1}{x(x+1)} dx = \int x + \frac{1}{x} - \frac{1}{x+1} dx = \frac{x^2}{2} + \ln|x| - \ln|x+1| + C$$

iv.

$$\int_{-1}^{3} (x-1)\sqrt{x+1}dx$$

Letting $u = x + 1, du = dx$, then $x - 1 = u - 2$. So,

$$\int_{-1}^{3} (x-1)\sqrt{x+1}dx = \int_{0}^{4} (u-2)\sqrt{u}du = \left(\frac{2u^{5/2}}{5} - \frac{4u^{3/2}}{3}\right)|_0^4 = 32/15$$

.

2. (20 pts) **Evaluate each of the following limits.**

i.

$$\lim_{x\to\pi/2} [\tan(x) - \sec(x))]$$

$$\lim_{x\to\pi/2} [\tan(x) - \sec(x))] = \lim_{x\to\pi/2} \left[\frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)}\right]$$

$$= \lim_{x\to\pi/2} \left[\frac{\sin(x) - 1}{\cos(x)}\right]$$

. Since this last limit is of the form (0/0) we can apply L'Hopital's Rule. Then

$$\lim_{x\to\pi/2} [\tan(x) - \sec(x))] = \lim_{x\to\pi/2} \left[\frac{\sin(x) - 1}{\cos(x)}\right]$$

$$= \lim_{x\to\pi/2} \left[\frac{\cos(x)}{-\sin(x)}\right] = 0.$$

ii.

$$\lim_{x\to 0}\left[(1+\sin x)^{\frac{2}{x}}\right]$$

Since this limit involves an exponential, we will first find

$$\lim_{x\to 0}\left[\ln\left[(1+\sin x)^{(}\frac{2}{x}\right]\right]=\lim_{x\to 0}\left[\frac{2\ln(1+\sin x)}{x}\right].$$

Since this last limit is of the form (0/0), applying L'Hopital's Rule we have

$$\lim_{x\to 0}\left[2\frac{\cos(x)}{(1+\sin(x))}\right]=2$$

. Then,

$$\lim_{x\to 0}\left[(1+\sin x)^{\frac{2}{x}}\right]=e^2$$

3.(20 pts) Evaluate each of the following improper integrals. If the integral diverges, explain why. If the integral converges, give a numerical value for the integral.

i.

$$\int_{1/3}^{+\infty}\frac{1}{1+9x^2}dx$$

Letting $u=3x, du=3dx$, we can rewrite the above integral as

$$(1/3)\int_{1}^{+\infty}\frac{1}{(1+u^2)}du=(1/3)\lim_{b\to+\infty}\int_{1}^{b}\frac{1}{(1+u^2)}du$$

$$=(1/3)\lim_{b\to+\infty}\left[\arctan(b)-\arctan(1)\right]=(1/3)[\pi/2-\pi/4]=\pi/12$$

.

ii.

$$\int_{0}^{6}\frac{1}{(x-3)^2}dx$$

Letting $u=x-3, du=dx$,

$$\int_0^6 \frac{1}{(x-3)^2}dx = \int_{-3}^0 \frac{1}{u^2}dx + \int_0^3 \frac{1}{u^2}dx$$

. Since neither of these integrals converge being similar to an improper intergral of the form $\int_0^1 \frac{1}{x^2}dx$, the given integral diverges.

4. (12 pts) Determine whether the following sequences converge or diverge. If the sequence diverges explain why. If the sequence converges, give the limit of the sequence.

i. $\{(\frac{-\pi}{10})^n\}$

Since$|(-\pi/10| < 1$, $\lim_{n\to+\infty} a_n = 0$.
ii. $\left\{\frac{\sqrt{n}}{\ln(n)}\right\}$

By L'Hopital's Rule, $\lim_{x\to+\infty} \frac{\sqrt{x}}{\ln(x)} = \lim_{x\to+\infty}\left[\left[\frac{1}{(2\sqrt{x})}\right]/[1/x]\right]$
$= \lim_{x\to+\infty}[\sqrt{x}/2] = +\infty$. Then the above sequence diverges.

5. (8 pts) Determine whether the following statements are True or False. If a statement is False, give a counterexample that shows why the statement is False.

i. If $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x)$, then $\lim_{x\to 0}\frac{f(x)}{g(x)} = 1$.

False: Let $f(x) = 3x, g(x) = 5(x)$. Then $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, yet, $\lim_{x\to 0}\left[\frac{f(x)}{g(x)}\right] = 3/5$.

ii. If $p > 0$ and $\int_1^{+\infty}\frac{1}{x^p}dx$ converges, then $\int_0^1 \frac{1}{x^p}dx$ must diverge.
bigskip
True

iii. If $\lim_{n\to+\infty}|a_n| = 5$, then $\lim_{n\to+\infty} a_n = 5$.

False: Consider $\{(-5)^n\}$. Here $lim_{n\to+\infty}|(-5)^n| = 5$, but $\{(-5)^n\}$ diverges.

iv. If $p > 0$ and $\int_1^{+\infty}\frac{1}{x^p}dx$ diverges, then $\int_0^1 \frac{1}{x^p}dx$ must converge.
False: If $p = 1$, both the $\int_1^{+\infty}\frac{1}{x^p}dx$ and the $\int_0^1 \frac{1}{x^p}dx$ diverge.