Midterm 1

Quiz Scores (out of 100): n = 139; mean = 79.5 25th percentile = 71; median (50th percentile) = 83; 75th percentile = 91.5

1.(21 pts) Give a numerical value for each of the following:

 $\ln(e^{\cos(\pi/2)}) = \cos(\pi/2) = 0$ $\arctan(\sin\pi/2) = \arctan(1) = \pi/4$ $\sin(\cos^{-1}(3/5)) = 4/5$

2. (21 pts) Find $\frac{dy}{dx}$ in each of the following: i. $y = x^2 \ln(3x + 2)$ $\frac{dy}{dx} = x^2(\frac{3}{3x+2}) + 2x \ln(3x + 2)$ ii. $y = \arctan(e^{3x})$. $\frac{dy}{dx} = \frac{1}{1+(e^{3x})^2} \frac{d(e^{3x})}{dx} = \frac{3e^{3x}}{1+e^{6x}}$ iii. $y = x^{\cos x}$ $\ln y = \ln(x^{\cos x}) = (\cos x)(\ln x)$ $\frac{1}{y} \frac{dy}{dx} = (\cos x)(1/x) + (\ln x)(-\sin x)$ $\frac{dy}{dx} = x^{\cos x}[(\cos x)(1/x) - (\ln x)(\sin x)]$

3.(21 pts) Evaluate the following integrals. Give a numerical value for the definite integral.

i.

ii.

$$\int \frac{1}{\sqrt{1-2x^2}} dx$$

Letting $u = \sqrt{2}x$, $du = \sqrt{2}dx$ we can rewrite the above integral as:

$$\frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{1-u^2}} du = \frac{1}{\sqrt{2}} \arcsin u + C = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2}x) + C$$
$$\int^{\ln 2} e^x$$

$$\int_0^{\ln 2} \frac{e^x}{2 + e^x} dx$$

Letting $u = 2 + e^x$ we have $du = e^x$. When $x = 0, u = 2 + e^0 = 3$ and when $x = \ln 2, u = 2 + e^{\ln 2} = 4$. So, the above integral becomes:

$$\int_{3}^{4} \frac{du}{u} = \ln u |_{3}^{4} = \ln 4 - \ln 3 = \ln(4/3)$$

iii.

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$$\int \sin^3(x) dx$$

We can rewrite the above integral in the form

$$\int \sin^2(x) \sin(x) dx$$

. Then using the identity, $\sin^2(x) = 1 - \cos^2(x)$, we have

$$\int \sin^2(x)\sin(x)dx = \int \sin(x)dx - \int \cos^2(x)\sin(x)dx$$

. If we let $u = \cos(x), du = -\sin(x)dx$ in the second integral, then we have,

$$\int \sin^3(x) dx = -\cos(x) + \frac{\cos^3(x)}{3} + C$$

4.(18 pts) Suppose that a bacteria population is growing in a culture in such a way that the rate of change in the number of bacteria is proportional to the amount of bacteria present. Initially there were 250 bacteria in the culture and 50 minutes later there were 600 bacteria present.

When will the number of bacteria in the culture reach 2000?

Given the condition that 'the bacteria population is growing in such a way that the rate of change in the number of bacteria is proportional to the amount of bacteria present', we know that the model for y, the number of bacteria present at time t, is $y = y_0 e^{kt}$. But $y_0 = y(0) = 250$, so we have: $y = 250e^{kt}$. Using the fact that y(50) = 600, we have the equation, $600 = 250e^{50k}$, or, $\ln[600/250] = 50k$. Then $k = \frac{\ln[600/250]}{50}$. So, the model for the particular situation described is:

$$y = 250e^{\frac{\ln[600/250]}{50}t}$$

Then setting y = 2000 in this model, we can solve for the time t as follows:

$$2000 = 250e^{\frac{\ln[600/250]}{50}t}$$

$$\ln[2000/250] = \frac{\ln[600/250]}{50}t$$

$$t = \frac{50 \ln[2000/250]}{\ln[600/250]}$$

- 5. (12 pts) Suppose that $g(x) = \int_0^x (t^2 + 1) dt$.
- i. What is g(3)? $g(3) = \int_0^3 (t^2 + 1)dt = (t^3/3 + t)|_0^3 = 12.$

ii. What is g'(x)? By the First Fundamental Theorem of Calculus, $g'(x) = x^2 + 1$

iii. The function g has an inverse function, g^{-1} . Explain why briefly. Since g'(x) > 0 for all x, g is a strictly increasing function and as a result has an inverse (Theorem A, Ch7.2).

iv. What is $(g^{-1})'(12)$?

From part i. above we know that g(3) = 12, so $g^{-1}(12) = 3$. Then using the Inverse Function Theorem (Theorem B, Ch 7.2), we know that $(g^{-1})'(12) = \frac{1}{g'(g^{-1}(12))} = \frac{1}{g'(3)} = \frac{1}{(3^2+1)} = \frac{1}{10}$.

6.(7 pts) Evaluate the following limit.

$$\lim_{n \to +\infty} \left(\frac{n+2}{n}\right)^n$$

Let t = n/2 and note that:

$$\lim_{n \to +\infty} \left(\frac{n+2}{n}\right)^n = \lim_{n/2 \to +\infty} \left(1 + \frac{1}{n/2}\right)^{(n/2)2} = \left(\lim_{t \to +\infty} \left(1 + \frac{1}{t}\right)^{(t)}\right)^2 = e^2$$