MATH 1260 - Test # 2 Sample Questions

Formula Sheet

- The **dot product** of two plane vectors $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ is
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2.$$
- The **dot product** of two space vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$
- The **cross product** of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is
$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle.$$
- The **projection** of a vector $\vec{u}$ onto a vector $\vec{v}$ is
$$\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2} \vec{v}.$$
- The **determinant** of a 2 by 2 matrix is
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$
- The **determinant** of a 3 by 3 matrix is
$$\det \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \vec{a} \cdot (\vec{b} \times \vec{c}).$$
- The **gradient** of a function $f(x, y)$ is
$$\nabla f = \langle f_x, f_y \rangle.$$
- The **gradient** of a function $f(x, y, z)$ is
$$\nabla f = \langle f_x, f_y, f_z \rangle.$$
- The **tangent plane** of a function $f(x, y)$ at a point $P = (a, b)$ is
$$z = f_x(P) \cdot (x - a) + f_y(P) \cdot (y - b) + f(P).$$
- **Chain Rule**
$$D(g \circ f)_P = Dg_{f(P)} \cdot Df_P.$$

(1) Calculate the following limits or show that they do not exist:

(a) $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$.

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^4}$.

(c) $\lim_{(x,y)\to(0,0)} \frac{x+2y^2}{3x^2+2y^2}$.

(d) $\lim_{(x,y)\to(0,0)} \frac{2x^3y^2}{x^6+y^4}$.

(e) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{3x^2+2y^2}$.

(f) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+3y^2}$.

(g) $\lim_{(x,y)\to(0,0)} \frac{3x^3+2\sqrt{y}}{x^2+y^2}$.

(h) $\lim_{(x,y)\to(0,0)} \frac{-3x^3-y^2}{3x^3+2y^2}$.

(i) $\lim_{(x,y)\to(0,0)} \frac{x^2y^5}{2x^4+3y^{10}}$.

(j) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^2+3y^4}$.

(2) Find a function $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(\vec{v}) = \vec{0}$ for every vector $\vec{v}$ in the plane of equation $2x-3y+z=0$.

(3) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,y) = 3x-2y$. Show that
(a) for every number c, the set of vectors in $\mathbb{R}^2$ mapping to c is a line;
(b) all the lines in part (a) are parallel.

(4) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = (2x-y, -x+2y)$. Find all vectors $\vec{v}$ such that $f(\vec{v}) = c\vec{v}$ for some number c.

(5) Let A be a 2 by 2 matrix. Show that, if there exists a non-zero vector $\vec{a}$ such that $\vec{a}\cdot A = \vec{0}$, then $\det(A) = 0$.

(6) Find all the numbers a and all non-zero vectors $\vec{v}$ such that

$$\vec{v} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = a\vec{v}.$$

(7) Let $f(x,y) = 2x^2y^3 - 3xe^y + x\ln(x) - \sin(xy)$.
(a) Find the partial derivatives f_x and f_y.
(b) Find the gradient of f at the point $(1,0)$.

(8) Find g_x, g_y, and g_z for the function $g(x,y,z) = (x+z^2)\cos(y)\sin(2y)$.

(9) Show that, if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function and P is a point, then the gradient $\nabla f(P)$ of f at P is equal to $\langle Df_{\langle 1,0\rangle}(P), Df_{\langle 0,1\rangle}(P)\rangle$. (Remember that $Df_{\vec{u}}(P)$ denotes the directional derivative of f at the point P in the direction $\vec{u}$)

(10) Consider the graph of the function $f(x,y) = 2x^2\ln(xy) + 3\sqrt{y}$. Find the tangent plane at the point $(1,1)$.

(11) Consider the functions

$$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \quad f(x_1,x_2,x_3) = \left(x_1^2x_3, 3x_1 + \frac{x_2}{x_3}\right)$$

$$g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad g(y_1,y_2) = \left(\sqrt{y_1}, 3y_1^2 - 2y_1e^{y_1y_2}\right).$$

Find the derivative of $g \circ f$ at the point $(1,-3,1)$.

(12) Let $f(x,y,z) = \left(x^2+y^2+z^2, \dfrac{9x}{x+y+z}\right)$ and $g(x,y) = (y-x, 2xy^2)$. Find the derivative of the composition $g \circ f$ at the point $(1,1,1)$.

(13) Let $z = f(x,y)$ be a function of two variables and let $x = (u^2-v^2)/2$ and $y = uv$ be change-of-variable equations. Calculate z_u and z_v in terms of f_x, f_y, u, and v.

(14) Let $f(x,y) = y + 3x^2$. Verify by direct calculation that the gradient vector $\nabla f(1,-1)$ is perpendicular to the level curve $f(x,y) = 2$ at $(1,-1)$. Sketch the level curve near $(1,-1)$ and the gradient vector at $(1,-1)$.

(15) (a) Find the stationary points of the function $f(x,y) = xy$ and classify them.
(b) Find the global maximum and minimum of $f(x,y)$ on the region $x^2+y^2 \leq 8$.

(16) (a) Find the stationary points of the function $f(x,y) = 2x^2+x+y^2-2$ and classify them.
(b) Find the global maximum and minimum of $f(x,y)$ in the region $x^2+y^2 \leq 4$.

(17) Let $f(x,y) = xy^2 - 5x^2y$. Find the maximum and minimum value of f on the region above the graph of $y = x^2$ and below the line $y = 4$.

(18) Your boss wants you to build a rectangular box. He gives you \$9 to buy the materials to build the box, and he wants the bottom to be stronger. The material for the bottom costs \$2/ft, and the material for the sides and the top costs \$1/ft. What are the dimensions of the box with the largest volume that you can build with the \$9 your boss gave you?

(19) A shipping box with two dividers is to be constructed from 48 m^2 of cardboard. What are the dimensions of the box with largest volume that can be constructed?

(20) Find the maximum volume of a rectangular box with no top that can be built with 1200 square feet of material.