Math 5310: Test 1

September 13, 2004

The first test will be on Sections 1.2, 1.3, 1.4, 1.5, 1.6, and 1.7.
Here are some problems to look at while you prepare for it.

1 Set Theory

From the book: Section 1.2, Numbers 8, 9, 10.

Problem 1.1. Prove or disprove: If $A \subset B \cup C$, then $A \subset B$ or $A \subset C$.

Problem 1.2. Prove that if $A \subset B$, then $A \cap B = A$.

Problem 1.3. Prove or disprove: If $A \cup B \neq A \cap C$, then $A \not\subset C$ or $B \not\subset A$.

2 Functions

From the book: Section 1.3, Numbers 6, 7, 27.

Problem 2.1. Let $f : S \to T$.
(a) Prove or disprove: If $A \subset S$ and f is 1-1, then $f^{-1}(f(A)) = A$.
(b) Prove or disprove: If $B \subset T$ and f is onto, then $f(f^{-1}(B)) = B$.

Problem 2.2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(n) = 2n + 3$.
(a) Prove that $f(\mathbb{Z})$ is the set of odd integers.
(b) Let $A = \{n \in \mathbb{Z} \mid -2 \leq n \leq 3\}$. Compute $f(A)$ (no formal proof required).

Problem 2.3. Let $f : S \to T$ be a surjective function and let A be a subset of S.
(a) Prove that $T - f(A) \subset f(S - A)$.
(b) Give an example to show that equality does not in general hold in part (a).

Problem 2.4. Let $f : S \to T$ and $g : T \to U$ be two functions.
Prove or disprove: If gf is surjective and g is injective, then f is surjective.

Problem 2.5. Let $f : S \to T$ be a function. Let $A \subset S$ and $B \subset T$.
(a) Prove that $f(A \cap f^{-1}(B)) = f(A) \cap B$.
(b) Prove or disprove: $f^{-1}(f(A) \cap B) \subset A \cap f^{-1}(B)$.
(c) Prove or disprove: $A \cap f^{-1}(B) \subset f^{-1}(f(A) \cap B)$.
(d) Prove or disprove: If f is 1-1, then $f^{-1}(B - f(A)) = f^{-1}(B) - A$.

3 $A(S)$

From the book: Section 1.4, Numbers 2, 3, 5, 8, 27, 28, 29.

4 The Integers

From the book: Section 1.5, Numbers 3, 4.

Problem 4.1. Prove that there are no integers x and y such that $x^2 = 8y + 3$.

Problem 4.2. Prove or disprove: if $r|s$ and $(s, t) = 1$, then $(r, t) = 1$.

Problem 4.3. Prove or disprove: if $r|s$ and $(r, t) = 1$, then $(s, t) = 1$.

Problem 4.4. Prove that if n is an odd integer, then $n^2 = 4q + 1$ for some integer q.

Problem 4.5. Prove that if a and b are not both odd, then $(a + b)^2 \equiv a^2 + b^2 (\text{mod } 4)$.

5 Induction

From the book: Section 1.6, Numbers 1, 8, 13.

Problem 5.1. Prove that for every positive integer n, $n^2 < 3^n$.

Problem 5.2. Prove that for every positive integer n, $3|(n^3 + 5n)$.

Problem 5.3. Prove that for every $n \geq 1$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq \frac{2n}{n+1}.$$

Problem 5.4. Prove that, for every positive integer n, $1^2 + 2^2 + 3^2 + \cdots + n^2 \leq (n-1)^3 + 6$.

6 Complex Numbers