

**Rank Two  
Vector Bundles  
on  
Irreducible  
Nodal Curves**

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## BACKGROUND

This work is about the moduli spaces of semi-stable vector bundles of rank 2 on an irreducible projective nodal curve  $C$  of (arithmetic) genus  $g$  over  $\mathbb{C}$ .

A vector bundle  $E$  is called semi-stable if

$$\frac{\deg F}{\operatorname{rk} F} \leq \frac{\deg E}{\operatorname{rk} E}$$

for every subbundle  $F \subseteq E$ .

For every pair of integers  $r > 0$  and  $d$ , there exists a moduli space of (S-equivalence classes of) semi-stable vector bundles of rank  $r$  and degree  $d$ :  $\mathcal{U}_C(r, d)$ .

Also, there exists a moduli space of semi-stable vector bundles of rank  $r$  and fixed determinant  $\Lambda^r E \simeq L$ , with  $L$  a line bundle:  $\mathcal{SU}_C(r, L)$ .

## SMOOTH CASE

In the cases  $g = 0$  and  $g = 1$ , these moduli spaces are completely understood, thanks to Grothendieck and Atiyah.

In general, these moduli spaces were constructed in the 60's by several people (D. Mumford, M.S. Narasimhan, S. Ramanan, C.S. Seshadri).

They are important because they are classifying spaces and because of their connection with Physics when  $L = \mathcal{O}_C$ .

The Picard group of  $SU_C(r, L)$  is isomorphic to  $\mathbb{Z}$  (J.M. Drezet, M.S. Narasimhan, 1989), and the linear systems of powers of the ample generator of  $\text{Pic}SU_C(r, \mathcal{O}_C)$  are directly related to conformal blocks studied in Conformal Field Theory.

## NODAL CASE

On a nodal curve  $C$ , the moduli spaces  $\mathcal{U}_C(r, d)$  and  $\mathcal{SU}_C(r, d)$  were constructed in the 70's by P.E. Newstead and C.S. Seshadri together with their natural compactifications  $\overline{\mathcal{U}_C(r, d)}$  and  $\overline{\mathcal{SU}_C(r, d)}$ .

These natural compactifications use torsion-free sheaves, i.e., sheaves that do not contain any subsheaf supported on a finite set of points.

The notion of semi-stability naturally extends to these sheaves.

# INTRODUCTION

We study the case of vector bundles of rank 2. In this case, a very useful tool used by many of the people mentioned before is that of extensions.

If  $L_1, L_2$  are line bundles, an extension of  $L_1$  by  $L_2$  is a short exact sequence

$$0 \longrightarrow L_2 \longrightarrow E \longrightarrow L_1 \longrightarrow 0.$$

The set of such extensions,  $\text{Ext}_C^1(L_1, L_2)$ , can be made into a vector space.

In his Ph.D. thesis (1989), Bertram analyzed the natural rational map

$$\phi_L: \mathbb{P}(\text{Ext}_C^1(L, \mathcal{O}_C)) \longrightarrow \mathcal{S}\mathcal{U}_C(2, L)$$

defined by

$$\phi_L(0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow L \rightarrow 0) = E,$$

and used it to describe  $\mathcal{S}\mathcal{U}_C(2, L)$  for a smooth curve  $C$ .

# NOTATION

Our object of study is the rational map

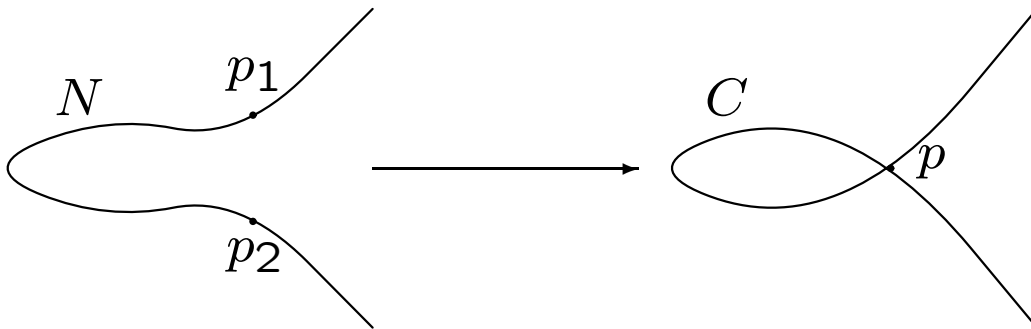
$$\phi_L: \mathbb{P}(\mathrm{Ext}_C^1(L, \mathcal{O}_C)) \longrightarrow \overline{\mathrm{SU}}_C(2, L)$$

defined by

$$\phi_L(0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0) = E.$$

$C$  is an irreducible nodal curve with a node  $p$ .

$\pi: N \rightarrow C$  is the normalization of  $C$ , with  $\pi^{-1}(p) = \{p_1, p_2\}$ :



# THE LOCUS OF INDETERMINANCY

$$\phi_L: \mathbb{P}(\mathrm{Ext}_C^1(L, \mathcal{O}_C)) =: \mathbb{P}_L \longrightarrow \overline{SU_C(2, L)}$$

$$\begin{array}{ccc} & \cup & \nearrow \\ & U_L & \end{array}$$

$$\phi_L(0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0) = E,$$

$$U_L = \{0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow L \rightarrow 0 \mid E \text{ is semi-stable}\}.$$

**Proposition.** (1) *If  $\deg L < 0$ , then  $U_L = \emptyset$ .*

(2) *If  $0 \leq \deg L \leq 2$ , then  $U_L = \mathbb{P}_L$ .*

(3) *If  $3 \leq \deg L \leq 4$ , then  $U_L = \mathbb{P}_L \setminus \varphi_{L \otimes \omega_C}(C)$ .*

(4) *If  $\deg L \geq 5$ , secant lines appears in the indeterminacy locus.*

We resolve the indeterminacy in case (3) via three blow-ups along smooth centers.

# THE FIRST BLOW-UP

$$\mathbb{P}_{L,1} := \mathcal{BL}_p \mathbb{P}_L \xrightarrow{\varepsilon_1} \mathbb{P}_L = \mathbb{P}(\mathrm{Ext}_C^1(L, \mathcal{O}_C)).$$

**Theorem.** *There is a commutative diagram of rational maps*

$$\begin{array}{ccc} E_1 \subseteq \mathbb{P}_{L,1} & & \\ \downarrow & \downarrow & \searrow \\ \{p\} \subseteq \mathbb{P}_L & \longrightarrow & \overline{SU_C(2, L)} \end{array}$$

The exceptional divisor  $E_1$  is canonically isomorphic to  $\mathbb{P}(\mathrm{Ext}_C^1(L, \pi_* \mathcal{O}_N))$ , and the image of a point  $x \in E_1$  corresponding to an extension  $E'_x$  in  $\mathrm{Ext}_C^1(L, \pi_* \mathcal{O}_N)$  is its image under the linear homomorphism

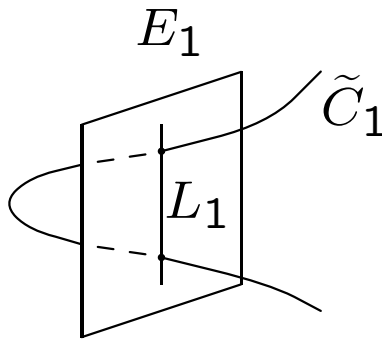
$$\mathrm{Ext}_C^1(L, \pi_* \mathcal{O}_N) \xrightarrow{\psi} \mathrm{Ext}_C^1(L \otimes (\pi_* \mathcal{O}_N)^*, \pi_* \mathcal{O}_N)$$

defined by pull-back:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_* \mathcal{O}_N & \longrightarrow & E'_x & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow & & \cdot \\ 0 & \longrightarrow & \pi_* \mathcal{O}_N & \longrightarrow & E_x & \longrightarrow & L \otimes (\pi_* \mathcal{O}_N)^* & \longrightarrow & 0 \end{array}$$

In particular, the indeterminacy locus of the rational map  $\mathbb{P}_{L,1} \rightarrow \overline{SU_C(2,L)}$  is the union of the strict transform  $\tilde{C}_1$  of  $C$  and the line

$$L_1 := \mathbb{P}(\ker \psi) \subseteq E_1.$$



The strict transform  $\tilde{C}_1$  of  $C$  is a copy of  $N$  that intersects  $E_1$  at the two points  $p_1, p_2$ , which are on  $L_1$ .

The points on  $L_1$  correspond to the directions in the tangent plane to  $C$  at  $p$ .

## THE SECOND BLOW-UP

$$\mathbb{P}_{L,2} := \mathcal{BL}_{L_1} \mathbb{P}_{L,1} \xrightarrow{\varepsilon_2} \mathbb{P}_{L,1} \xrightarrow{\varepsilon_1} \mathbb{P}_L.$$

**Theorem.** *There is a commutative diagram of rational maps*

$$\begin{array}{ccc} E_2 \subseteq \mathbb{P}_{L,2} & & \\ \downarrow & \downarrow & \searrow \\ L_1 \subseteq \mathbb{P}_{L,1} & \longrightarrow & \overline{SU_C(2, L)} \end{array}$$

and for each  $l \in L_1$ , the map

$$E_2|_l \longrightarrow \overline{SU_C(2, L)}$$

is given by a linear map

$$\mathcal{N}_l \longrightarrow H',$$

where

$$\mathcal{N}_l = \mathcal{N}_{L_1/\mathbb{P}_{L,1}}|_l$$

and

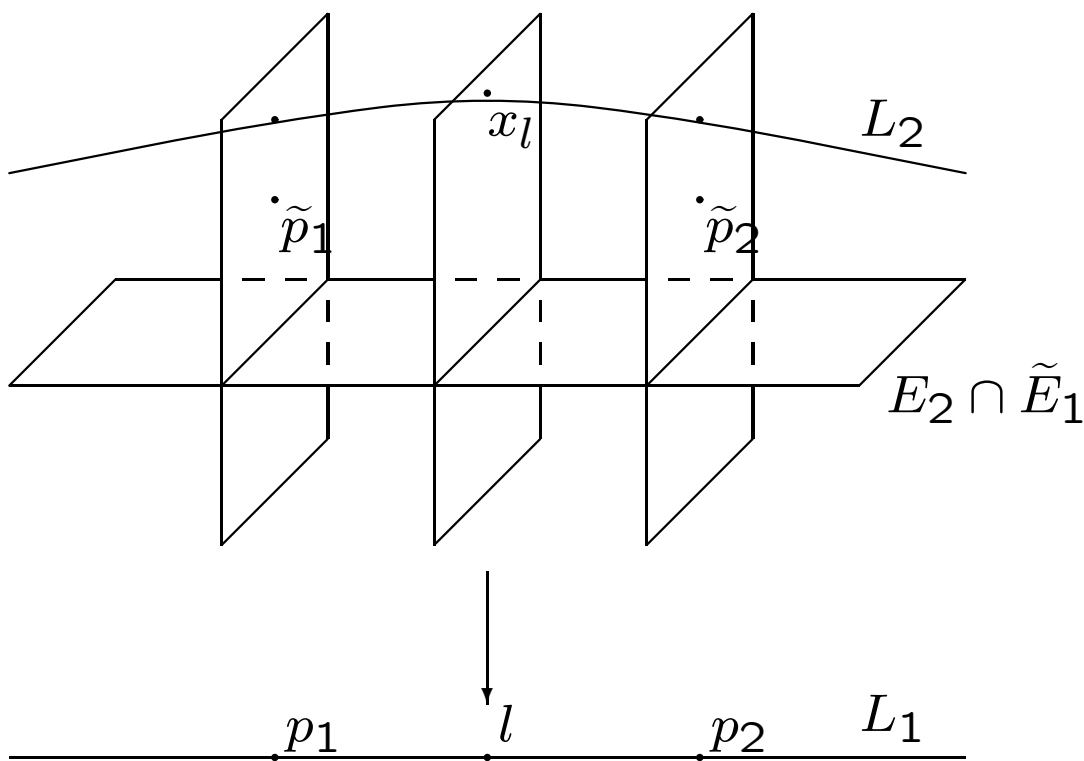
$$H' = \overline{\{\det E \simeq L\}} \subseteq \text{Ext}_C^1(L \otimes (\pi_* \mathcal{O}_N)^*, \pi_* \mathcal{O}_N).$$

The linear map is an isomorphism if  $l \neq p_1, p_2$ , and it maps  $\mathcal{N}_{p_i}$  ( $i = 1, 2$ ) surjectively onto  $\text{Im}\psi \subseteq H'$ , where

$$\text{Ext}_C^1(L, \pi_*\mathcal{O}_N) \xrightarrow{\psi} \text{Ext}_C^1(L \otimes (\pi_*\mathcal{O}_N)^*, \pi_*\mathcal{O}_N).$$

In particular, the indeterminacy locus of the rational map  $\mathbb{P}_{L,2} \rightarrow \overline{SU_C(2, L)}$  is the strict transform  $\tilde{C}_2$  of  $\tilde{C}_1$ .

Picture of  $E_2$ :



## THE THIRD BLOW-UP

$$\mathbb{P}_{L,3} := \mathcal{BL}_{\tilde{C}_2} \mathbb{P}_{L,2} \xrightarrow{\varepsilon_3} \mathbb{P}_{L,2} \xrightarrow{\varepsilon_2} \mathbb{P}_{L,1} \xrightarrow{\varepsilon_1} \mathbb{P}_L.$$

**Theorem.** *There is a commutative diagram of rational maps*

$$\begin{array}{ccc} E_3 \subseteq \mathbb{P}_{L,3} & & \\ \downarrow & \downarrow & \searrow \\ \tilde{C}_2 \subseteq \mathbb{P}_{L,2} & \longrightarrow & \overline{SU_C(2, L)} \end{array}$$

and the map

$$\mathbb{P}_{L,3} \longrightarrow \overline{SU_C(2, L)}$$

is a morphism. For each point  $q \in \tilde{C}_2$  that maps to a smooth point  $q \in C$ , the morphism

$$E_3|_q \longrightarrow \overline{SU_C(2, L)}$$

is given by a linear isomorphism

$$\mathcal{N}_{\tilde{C}_2/\mathbb{P}_{L,2}}|_q \longrightarrow \text{Ext}_C^1(L(-q), \mathcal{O}_C(q)).$$

For the two points  $\tilde{p}_1, \tilde{p}_2 \in \tilde{C}_2$  that map to the singular point  $p \in C$ , the morphism

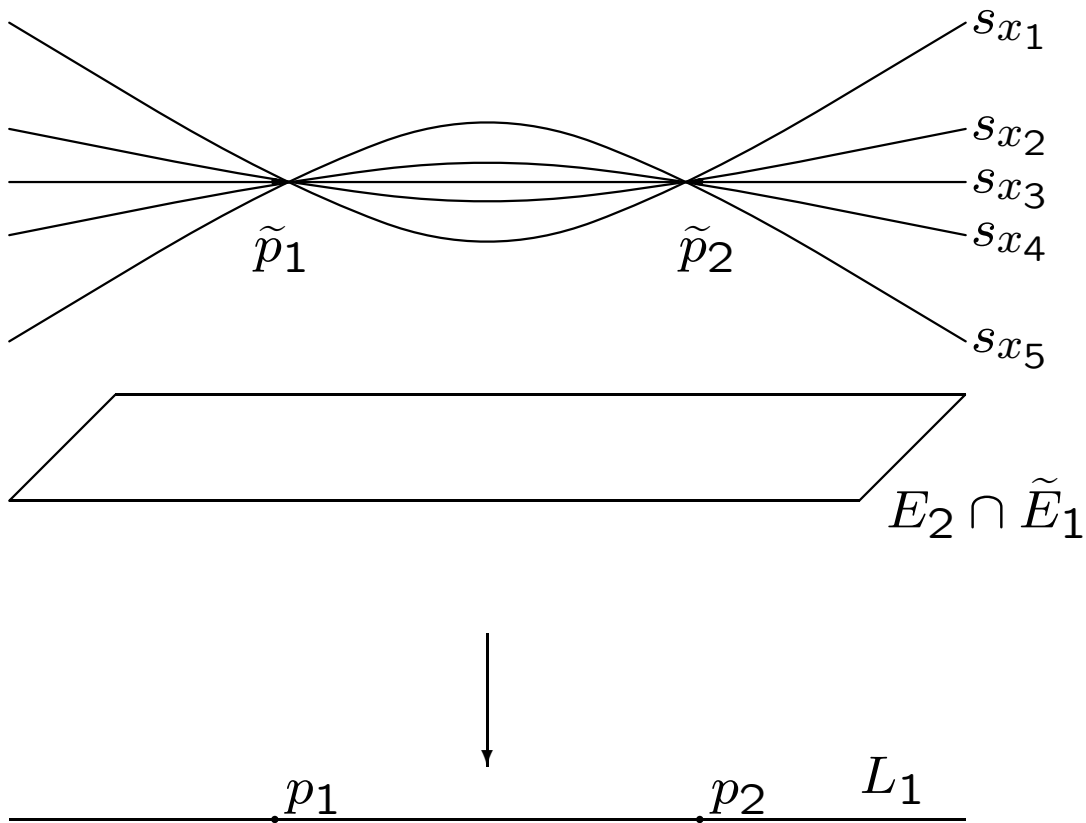
$$E_3|_{\tilde{p}_i} \longrightarrow \overline{\mathcal{SU}_C(2, L)} \quad (i = 1, 2)$$

is given by a linear isomorphism

$$\mathcal{N}_{\tilde{C}_2/\mathbb{P}_{L,2}}|_{\tilde{p}_i} \longrightarrow H',$$

where

$$H' = \overline{\{\det E \simeq L\}} \subseteq \text{Ext}_C^1(L \otimes (\pi_* \mathcal{O}_N)^*, \pi_* \mathcal{O}_N).$$



## IDEA OF THE PROOF

At each step, we construct a universal sheaf  $\mathcal{E}_{L,i}$  on  $\mathbb{P}_{L,i} \times C$  such that, for every  $x \in \mathbb{P}_{L,i}$  where the map is defined,  $\mathcal{E}_{L,i}|_{\{x\} \times C}$  is the image of  $x$  in  $\overline{\mathcal{SU}_C(2, L)}$ .

On each fiber of the exceptional divisors, the map is induced by a linear homomorphism

$$\mathcal{N}|_x \longrightarrow \text{Ext}_C^1(F, G),$$

where  $x$  is a point on the blown-up locus, and  $F, G$  are torsion-free sheaves on  $C$ .

We prove this in two steps.

(1) We show that the universal sheaf  $\mathcal{E}_{L,i}$  fits in a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{N}|_x)}(1) \boxtimes G \longrightarrow \mathcal{E}_{L,i}|_{\mathbb{P}(\mathcal{N}|_x) \times C} \longrightarrow \pi_C^* F \longrightarrow 0$$

when restricted to  $\mathbb{P}(\mathcal{N}|_x) \times C$ .

(2) We prove that

$$\begin{aligned} & \text{Ext}_{\mathbb{P}^N \times C}^1(\pi_C^* F, \mathcal{O}_{\mathbb{P}^N}(1) \boxtimes G) \\ & \simeq H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \otimes \text{Ext}_C^1(F, G) \\ & \simeq \text{Hom}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))^*, \text{Ext}_C^1(F, G)). \end{aligned}$$

The result follows from the fact that

$$H^0(\mathbb{P}(\mathcal{N}|_x), \mathcal{O}_{\mathbb{P}(\mathcal{N}|_x)}(1))^* \simeq \mathcal{N}|_x.$$

# IMAGE OF THE MORPHISM

After the first blow-up, the image of

$$\mathbb{P}_{L,1} \longrightarrow \overline{SU_C(2, L)}$$

“adds” the torsion-free sheaves of the form

$$0 \longrightarrow \pi_* \mathcal{O}_N \longrightarrow E \longrightarrow L \otimes (\pi_* \mathcal{O}_N)^* \longrightarrow 0$$

that are push-forwards from  $N$ .

After the second blow-up, the image of

$$\mathbb{P}_{L,2} \longrightarrow \overline{SU_C(2, L)}$$

“adds” all the torsion-free sheaves of the same form that have determinant  $L$ .

After the third blow-up, the image of

$$\mathbb{P}_{L,3} \longrightarrow \overline{SU_C(2, L)}$$

“adds” all the torsion-free sheaves of the form

$$0 \longrightarrow \mathcal{O}_C(q) \longrightarrow E \longrightarrow L(-q) \longrightarrow 0$$

for some smooth point  $q \in C$ .

In general, we can prove the following.

**Proposition.** *A torsion-free sheaf  $E \in \overline{SU_C(2, L)}$  is in the image of the morphism*

$$\mathbb{P}_{L,3} \longrightarrow \overline{SU_C(2, L)}$$

*if and only if  $H^0(E) \neq 0$ .*

## FIBERS OF THE MORPHISM

**Proposition.** *The morphism  $\mathbb{P}_{L,3} \rightarrow \overline{SU_C(2, L)}$  has connected fibers.*

If  $g = 2$  and  $\deg L = 3$ , then the rational map  $\mathbb{P}_L \rightarrow \overline{SU_C(2, L)}$  is birational, and we obtain the following corollary.

**Corollary.** *If  $g = 2$  and  $\deg L = 3$ , then the normalization morphism*

$$\overline{SU_C(2, L)}^\nu \longrightarrow \overline{SU_C(2, L)}$$

*is one-to-one.*