RESEARCH STATEMENT

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1. Introduction

Over the last twenty years, there has been a great deal of interaction between mathematics and physics, to the mutual advantage of both. In particular, I would like to focus on how ideas inspired by physics have contributed to several new 'discoveries' in algebraic geometry in the recent decades, and how conjectures from algebraic geometry have contributed to the development of new physical theories. The physical theories I am referring to are closely related to several moduli spaces that naturally appear in algebraic geometry.

For example, Yang-Mills gauge theories play an important role both in physics, where they describe the physics of particle interactions, and in mathematics, where they are really important in the studying of the moduli spaces of vector bundles on a curve. In the late '80s and early '90s, a great deal of thought went into finding a mathematical proof for the surprising Verlinde formula, which gave a prediction for the dimension of the space of sections of a natural line bundle on these moduli spaces (for a survey on this, see [Bea95]). Besides improving the mathematical understanding of the moduli spaces of vector bundles on a curves, other moduli spaces had to be introduced and studied, like the moduli space of parabolic bundles or the moduli space of stable pairs.

Another example is mirror symmetry, which is of great interest to both physicists and mathematicians (a reference is [MirSym03]). Mirror symmetry started in string theory from the observation that certain worldsheet theories did not uniquely determine the corresponding manifold, but determined a pair of manifolds instead. These two manifolds were called "mirrors" of each other, and several predictions were made on how they relate to each other. Just as for the Verlinde formula, a lot of effort went into finding a mathematical proof of these predictions, except that in this case mathematicians even had to first find a precise mathematical statement. Mirror symmetry gives a relation between certain (relatively easy to calculate) invariants of a manifold and other (hard to calculate) invariants on the "mirror" manifold, and this has been used to solve many enumerative questions in algebraic geometry. An example of these invariants are the Gromov-Witten invariants, and their study led to a better understanding of the moduli space of curves. Moreover, just as in the previous example, new moduli spaces had to be introduced, like the moduli space of stable maps or the moduli space of admissible covers.

Several proofs and new conjectures arose from the predictions of mirror symmetry, and the new conjectures helped in the development of new physical theories. For example, a conjecture by Kontsevich in [Kon95], inspired by mirror symmetry, predicted an equivalence between two categories, the derived category of coherent sheaves on a manifold, and the Fukaya category of the "mirror." This caught physicists by surprise, and new theories have been developed to find a physical explanation for this conjecture. This is related to the concept of π-stability for objects in the derived category, and after Bridgeland gave a precise corresponding definition of stability condition in [Bri1], several algebraic geometers started studying moduli spaces of stability conditions.

My research has involved several of the moduli spaces above. My first results were about the moduli spaces of vector bundles on curves, both extending some known results to the case of nodal curves (see [Arc]) and proving new results trying to answer open questions raised in [Bea95] (see [Arc05]).

In studying the moduli spaces of curves, one important construction is the tautological ring, which is a subring of the Chow ring which contains all natural classes arising from the geometry of these moduli spaces. There is a nice conjectural description of these tautological rings (see [Fab99]), and a lot of work has been put into finding relations among its known generators. In distinct joint works, with Y.-P. Lee (see [ArcLee1] and [ArcLee2]) and with F. Sato (see [ArcSat]), we have found new relations via techniques that allow to find many relations in a uniform way.

Derived categories have started to play a greater role in algebraic geometry in the past few years, and, as I mentioned above, several people have now been studying moduli spaces of stability conditions. In joint work with A. Bertram (see [ArcBer]), we took the next step, and we started studying moduli spaces of stable objects in the derived category, i.e., instead of studying a space classifying stability conditions, we study a space classifying stable objects for a given stability condition.

In the next few section, I shall give more details about the moduli spaces mentioned above.

2. Moduli Spaces of Stable Objects in the Derived Category

2.1. **Background.** Let S be an algebraic K3 surface over $\mathbb{C}$, i.e., an algebraic surface such that $\omega_S \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. Assume that $\mathrm{Pic} S \simeq \mathbb{Z}$, generated by an ample line bundle $\mathcal{O}_S(1)$. If we let $H := c_1(\mathcal{O}_S(1))$, then $H^2 = 2g - 2$, where $g \geq 2$ is, by definition, the genus of S.

Let $\mathcal{D}(S)$ be the bounded derived category of coherent sheaves of $\mathcal{O}_S$-modules. Bridgeland defined a concept of stability condition for $\mathcal{D}(S)$, and studied the case of K3 surfaces in [Bri2].

For every stability condition $\mathcal{P}$ and every triple (r, c_1, ch_2), one would like to define a moduli space $\mathcal{M}_{\mathcal{P}}(r, c_1, \mathrm{ch}_2)$ of objects $E \in \mathcal{D}(S)$ which are stable for $\mathcal{P}$ and have invariants $\mathrm{rk} E = r$, $c_1(E) = c_1$, and $\mathrm{ch}_2(E) = \mathrm{ch}_2$. These moduli spaces generalize the moduli spaces of torsion-free sheaves on S. Indeed, for certain values of $\mathcal{P}$ and (r, c_1, ch_2), the only stable objects are torsion-free sheaves, where stability for torsion-free sheaves is defined using the slope $\mu_H(E) := (c_1(E) \cdot H)/\mathrm{rk} E$.

2.2. **Our results (joint work with A. Bertram).** We study a 1-dimensional family inside the space of stability conditions, depending on a real parameter $\alpha > 0$. The family is chosen in such a way that, for all α, the objects with invariants $(0, H, g-1)$ have phase $1/2$, where the phase is the invariant in the derived category considered by Bridgeland to define stability. To be a little more precise, there exists functions Z_α defined on an abelian subcategory $\mathcal{A}$ of $\mathcal{D}(S)$, and the phase of an non-zero object $E \in \mathcal{A}$ is the number $\phi_\alpha(E)$ such that $Z_\alpha(E) = m_E e^{i\pi\phi_\alpha(E)}$ for some $m_E \in \mathbb{R}_{>0}$. An object $E \in \mathcal{A}$ is said to be α-stable if $\phi_\alpha(F) < \phi_\alpha(E)$ for every proper subobject $F \subseteq E$ in $\mathcal{A}$.

Our main theorem can be summarized as follows.

Theorem 2.1 (A.-Bertram). *The moduli spaces $\mathcal{M}_\alpha(0, H, g-1)$ exist for $\alpha > (g+3)/36$, and they are smooth proper symplectic varieties. Moreover, there are only finitely many distinct such moduli spaces, and they are related via Mukai flops*

$$\begin{array}{ccccccccccccc}
 & & & \widetilde{\mathcal{M}}_{\alpha_2} & & & & \widetilde{\mathcal{M}}_{\alpha_1} & & & & \widetilde{\mathcal{M}}_{\alpha_0} & \\
\cdots & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow \\
 & \mathcal{M}_{\alpha_3} & & & & \mathcal{M}_{\alpha_2} & & & & \mathcal{M}_{\alpha_1} & & & \mathcal{M}_{\alpha_0} \\
 & \bigcup & & & & \bigcup & & & & \bigcup & & & \bigcup \\
 & \mathbb{P}_3, \mathbb{P}_2^\vee & & & & \mathbb{P}_2, \mathbb{P}_1^\vee & & & & \mathbb{P}_1, \mathbb{P}_0^\vee & & & \mathbb{P}_0
\end{array}$$

where each of the $\mathbb{P}_i$'s is a projective bundle over $\mathrm{Hilb}^i(S) \times \mathrm{Hilb}^i(S)$ *with fibers isomorphic to* $\mathbb{P}(\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{I}_Z(1), \mathcal{I}_W^\vee[1]))$ *over a point* (Z, W).

We also conjecture that they are actually projective varieties, but we have not had the time to check it yet. The main ingredient of the proof is the characterization of the stable objects for a given value of α. We also prove that, as expected, for $\alpha >> 0$, the moduli space $\mathcal{M}_\alpha(0, H, g-1)$ is isomorphic to the relative Picard variety $\mathrm{Pic}_{2g-2} \to |\mathcal{O}_S(1)|$ of torsion-free sheaf of rank 1 and of degree $2g-2$ supported on curves $C \in |\mathcal{O}_S(1)|$.

2.3. **Future developments.** The original purpose of studying this problem was to study Thaddeus-type flips for a surface embedded in projective space. For a smooth curve C, and a line bundle L on C, Thaddeus constructed in [Tha94] moduli spaces of stable pairs. These moduli spaces are related to each other via flips, and at one end of the spectrum we encounter the projective space $\mathbb{P}(H^0(C, L)^\vee)$. If L is very ample, then C embedds into this projective space, and the first flip is a blow-up of the curve followed by a blow-down, while the other flips are related to blow-ups of the strict transforms of higher secant varieties of the embedded curve.

In our case, the K3 surface S is embedded into $\mathbb{P}_0^\vee = \mathbb{P}(H^0(S, \mathcal{O}_S(1))^\vee)$ (as long as $g \geq 3$, which we would assume for this part), and we expect the Mukai flops of Theorem 2.1, starting with the second flop, since $\mathbb{P}_0^\vee$ appears after the first flop as a subspace of $\mathcal{M}_{\alpha_1}$, to restrict to Thaddeus-type flips on $\mathbb{P}_0^\vee$. We checked the first such flip, and it does involve blowing-up the embedded surface.

Other future topics of research are:

- Use the same technique for other moduli spaces $\mathcal{M}(r, c_1, \mathrm{ch}_2)$.
- Extend these results to other surfaces.
- Use these techniques to prove the equality between Donaldson and Seiberg-Witten invariants.

3. Moduli Spaces of Curves and their Tautological Rings

3.1. **Background.** Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves of genus g with n marked points, where a curve C is called stable if it has at most nodes as singularities and it has a finite group of automorphisms.

Since the Chow ring of $\overline{\mathcal{M}}_{g,n}$ is very complicated in general, people have been studing a subring called the tautological ring. It is a ring that contains all natural geometric classes, and that is closed under pull-backs and push-forwards via the following three natural maps:

- The 'forgetful' map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ which forgets a marking.
- The map $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$ which glues two marked points into a node.
- The boundary map $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+2} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ which glues two points on two disconnected stable curves into a node.

Some examples of tautological classes are the ψ-classes, the λ-classes, and the κ-classes. The tautological ring has a very nice conjectural description [Fab99]. A set of generators is known, and my work has been focusing in finding new relations between these generators. To do it, I used two different techniques, one developed by Givental and Lee, and the other, the localization theorem, used in many areas by many people, and adapted to this situation as the virtual localization theorem by Graber and Pandharipande.

3.2. **Our results I (joint work with Y.-P. Lee).** Y.-P. Lee conjectured that there exists a finite algorithm which can calculate all of the relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$. It is a recursive algorithm, in the sense that, for a fixed pair (g, n), to find tautological relations in $\overline{\mathcal{M}}_{g,n}$, one needs to know all tautological relations in $\overline{\mathcal{M}}_{g',n'}$ with $g' \leq g$, and $n' \leq N(g, g')$, where $N(g, g')$ is a number that only depends on g and g'.

This conjecture was first proved to be true in genus 0 and genus 1 by Givental and Lee, and then in genus 2 for $n \leq 3$ in [ArcLee1], i.e., we used the conjecture to rederive all previously known tautological relation in a uniform way. The important part of this first result is that we were able to produce all of these relations with the same method, while their original individual proofs were very different.

We then used the conjecture to find a relation for ψ^3 in $\overline{\mathcal{M}}_{3,1}$ in [ArcLee2].

3.3. **Our results II (joint work with F. Sato).** Mumford proved in [Mum83] that $\psi^g - \lambda_1\psi^{g-1} + \cdots + (-1)^g\lambda_g = 0$ in the Chow ring of the moduli space of smooth curves $\mathcal{M}_{g,1}$. Sato and I found an explicit recursive formula for $\psi^g - \lambda_1\psi^{g-1} + \cdots + (-1)^g\lambda_g$ in the tautological ring of $\overline{\mathcal{M}}_{g,1}$ as a combination of boundary strata.

Theorem 3.1 (A.-Sato). *In the tautological ring of* $\overline{\mathcal{M}}_{g,1}$,

$$\sum_{i=0}^{g}(-1)^i\lambda_i\psi^{g-i} = \sum_{h=1}^{g-1}\left(\frac{h}{g} - 1\right)\iota_{h*}c_h,$$

where

$$c_h := \sum_{i=0}^{g-1}(-1)^{h+i+1}\left[\left(\sum_{j=0}^{h}(-1)^j\lambda_j^0\psi_0^{i-j}\right)\left(\sum_{j=0}^{g-h}(-1)^j\lambda_j^\infty\psi_\infty^{g-1-i-j}\right)\right],$$

ι_h is the natural boundary map $\iota_h\colon \overline{\mathcal{M}}_{h,2} \times \overline{\mathcal{M}}_{g-h,1} \to \overline{\mathcal{M}}_{g,1}$, ψ_0, ψ_∞ are descendents at the marked points glued by ι_h, and λ^0, λ^∞ are λ-classes on $\overline{\mathcal{M}}_{h,2}$ and $\overline{\mathcal{M}}_{g-h,1}$, respectively.

The nice thing about this formula is that it is recursive. The c_h classes supported on the boundary are a product of two classes on lower genus curves, one of which can be calculated via the the same formula. We explicitly calculated the answer for $g \leq 5$ [1].

[1]The answer was already known for $g = 1$ and $g = 2$, but unknown for higher g's.

3.4. **Future developments.** I am currently working with an undergraduate student on writing a computer program which would implement the algorithm of Y.-P. Lee's conjecture. If successful, we should be able to use the computer program to find new relations. We also would like to find the Betti numbers for the moduli spaces $\overline{\mathcal{M}}_{g,n}$ for low g's and n's.

As for the joint work with F. Sato, we think that Theorem 3.1 above is just the first step of an algorithm that should calculate each of the classes ψ^g, $\lambda_1\psi^{g-1}$, …, $\lambda_{g-1}\psi$, λ_g in terms of boundary strata. Graber and Vakil proved in [GraVak] that the classes ψ^g, $\lambda_1\psi^{g-1}$, …, $\lambda_{g-1}\psi$, λ_g are supported on the boundary strata with at least one genus 0 component[2]. Our algorithm, if successful, would complement their result by producing an explicit formula.

Conjecture (A.-Sato)**.** *There exists a finite algorithm which explicitly calculates the classes ψ^g, $\lambda_1\psi^{g-1}$, …, $\lambda_{g-1}\psi$, λ_g in the tautological ring of $\overline{\mathcal{M}}_{g,1}$ in terms of classes supported on boundary strata.*

We can prove the conjecture for $g \leq 3$, and we are working on a general proof for all g's.

4. Moduli Spaces of Torsion-free Shaves on Irreducible Curves

4.1. **Background.** Let C be a smooth irreducible projective curve of genus $g \geq 2$, and let $\mathcal{S}U_C(r, L)$ be the moduli space of semi-stable vector bundles of rank r and determinant L, where a vector bundle E on C is called semi-stable if, for every vector subbundle $F \subseteq E$, $\deg(F)\mathrm{rk}(E) \leq \deg(E)\mathrm{rk}(F)$.

The Picard group of $\mathcal{S}U_C(r, L)$ is generated by an ample line bundle $\mathcal{L}_r$ [DreNar89]. When $L = \mathcal{O}_C$, every divisor Θ_r on $\mathcal{S}U_C(r, \mathcal{O}_C)$ such that $\mathcal{O}_{\mathcal{S}U_C(r,\mathcal{O}_C)}(\Theta_r) \simeq \mathcal{L}_r$ is called a "generalized theta divisor." Beauville, Narasimhan, and Ramanan proved in [BeaNarRam89] that $|\Theta_r|^*$ is isomorphic to the linear system $|r\Theta|$, where Θ is the canonical theta divisor in $\mathrm{Pic}^{g-1}(C)$ defined by $\Theta := \{L \in \mathrm{Pic}^{g-1}(C) \mid H^0(C, L) \neq 0\}$. The induced (rational!) map $\phi_{\mathcal{L}_r} \colon \mathcal{S}U_C(r, \mathcal{O}_C) \to |r\Theta|$ is then defined by $\phi_{\mathcal{L}_r}(E) = \{L \in \mathrm{Pic}^{g-1}(C) \mid H^0(C, E \otimes L) \neq 0\}$. The base locus of $|\Theta_r|$ is the set of vector bundles $E \in \mathcal{S}U_C(r)$ such that $H^0(C, E \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^{g-1}(C)$.

4.2. **My Results.** I summarize here my Ph.D. Thesis and the two papers [Arc05] and [Arc].

4.2.1. *Extension Spaces and $\mathcal{S}U_C(2, L)$.* Bertram used the extension spaces $\mathrm{Ext}^1_C(L, \mathcal{O}_C)$ to study $\mathcal{S}U_C(2, L)$ in his Ph.D. Thesis [Ber89],[Ber92]. He studied the rational map ϕ_L from $\mathbb{P}(\mathrm{Ext}^1_C(L, \mathcal{O}_C))$ to $\mathcal{S}U_C(2, L)$ defined by $\phi_L(0 \to \mathcal{O}_C \to E \to L \to 0) = E$, and found several applications.

In my Ph.D. Thesis and [Arc], I extended his results to an irreducible projective curve C with only nodes as singularity, and I used it to study the compactification $\overline{\mathcal{S}U_C(2, L)}$ of $\mathcal{S}U_C(2, L)$ defined using torsion-free sheaves (see [New78] and [Ses82]).

Theorem 4.1 (A.)**.** *Let C be an irreducible projective nodal curve of arithmetic genus ≥ 2, and let L be a generic line bundle on C of degree 3 or 4. Let $\phi_L \colon \mathbb{P}(\mathrm{Ext}^1_C(L, \mathcal{O}_C)) \to \mathcal{S}U_C(2, L) \subseteq \overline{\mathcal{S}U_C(2, L)}$ be the natural rational map defined by $\phi_L([0 \to \mathcal{O}_C \to E \to L \to 0]) = E$. There exist a sequence of three blow-ups with smooth centers*

$$X_3 \xrightarrow{\varepsilon_3} X_2 \xrightarrow{\varepsilon_2} X_1 \xrightarrow{\varepsilon_1} \mathbb{P}(\mathrm{Ext}^1_C(L, \mathcal{O}_C)),$$

[2] It is easy to check this for the class $\psi^g - \lambda_1\psi^{g-1} + \cdots + (-1)^g\lambda_g$ with the recursive formula we obtained.

such that $\phi_L \circ \varepsilon_1 \circ \varepsilon_2 \circ \varepsilon_3 \colon X_3 \longrightarrow \overline{\mathcal{S}U_C(2,L)}$ extends to a morphism $\phi_{L,3}$. Moreover, there exist a universal sheaf $\mathcal{E}$ on $X_3 \times C$ realizing the morphism $\phi_{L,3}$.

As in the smooth case, the curve C embeds into $\mathbb{P}(\operatorname{Ext}^1_C(L, \mathcal{O}_C))$. The first blow-up is the blow-up at the nodes of C. The second blow-up is the blow-up of the lines in the exceptional divisor of the first blow-up that corresponds to lines through a node p contained in T_pC, the two-dimensional tangent space to C at p. The center of the third blow-up is the strict transform of the curve C.

Among the several applications, let me point out that ϕ_L can be used to prove that the results for a smooth curve of genus 1 (see [Ati57] and [Tu93]) generalize to the irreducible nodal case, and that using ϕ_L for a line bundle L of high enough degree, we can prove the following statement, which generalized the same statement for $A_{3g-4}(\mathcal{S}U_C(2,L))$ proved by Bhosle (see [Bho99] and [Bho04]).

Corollary 4.2 (A.). *If C is an irreducible nodal curve of genus ≥ 2, then*

$$A_{3g-4}(\overline{\mathcal{S}U_C(2,L)}) \simeq \mathbb{Z}.$$

4.2.2. *The Base Locus of the Linear Systems of Generalized Theta Divisors.* There are many open questions in this area (for a survey, see [Bea95] and the more recent [Bea]). Let C be a smooth irreducible projective curve of genus $g \geq 2$. Raynaud proved in [Ray82] the following theorem[3].

Theorem (Raynaud). (*a*) *For $r = 2$, the linear system $|\Theta_2|$ has no base points.*
(*b*) *For $r = 3$, $|\Theta_3|$ has no base points if $g = 2$, or if $g \geq 3$ and C is generic.*
(*c*) *Let n be an integer ≥ 2 dividing g. For $r = n^g$, the linear system $|\Theta_r|$ has base points.*

Using the dual E_L of the kernel of the evaluation map e_L for a line bundle L generated by its global sections, and its exterior powers, Popa [Pop99] and Schneider [Sch] proved the existence of other vector bundles in the base locus of the generalized theta divisor.

In particular, Schneider defines a condition (R) as follows: A vector bundles E has the property (R) if, for every $n \in \mathbb{Z}$ and any generic line bundle L of degree n, $H^0(E \otimes L) = 0$ or $H^1(E \otimes L) = 0$. He then proves that, if L is a line bundle of degree $\geq 2g+2$, then $\Lambda^p E_L$ does not verify (R) for every $2 \leq p \leq \operatorname{rk} E_L - 2$. Since all such $\Lambda^p E_L$ are semi-stable, whenever the slope of $\Lambda^p E_L$ is integral, this easily produces examples of vector bundles in the base locus of the generalized theta divisor.

As our first result, we prove that every vector bundle without the property (R) "produces" a vector bundle in the base locus of the generalized theta divisor, hence making it possible to use all of the bundles studied by Raynaud, Popa, and Schneider, even the ones with non-integral slope.

Theorem 4.3 (A.). *If E is a semi-stable vector bundle of rank r on C which does not satisfy the property (R), then the base locus of $|\Theta_r|$ is non-empty.*

Using Raynaud's and Schneider's results, we obtain the following corollary.

Corollary 4.4 (A.). *The base locus of $|\Theta_r|$ is non-empty for $r = 2^g$ and $r = (g+1)(g+2)/2$.*

[3] I state it here as stated by Beauville in [Bea95].

As Popa points out in [Pop99], this implies that the base locus is also non-empty for any bigger rank. If we let r_0 be the lowest rank such that the base locus of $|\Theta_r|$ is non-empty, the corollary above can be restated as

$$r_0 \leq \min\left\{2^g, \frac{(g+1)(g+2)}{2}\right\}.$$

We produce the following lower bound for the dimension of the base locus of $|\Theta_r|$:

Proposition 4.5 (A.). *Let C be a smooth complex projective curve of genus g. Then the dimension of the base locus of $|\Theta_r|$ is at least $(r-r_0)^2(g-1)+1$, where r_0 is the minimum rank for which the base locus of the generalized theta divisor is non-empty.*

Many questions are still unanswered (see [Bea]). The upper bound for r_0 found above is probably not sharp. Also, while all the bundles in the base locus of $|\Theta_{r_0}|$ are stable, no stable bundles are (to my knowledge) known to exist in the base locus of $|\Theta_r|$ if $r > r_0$.

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