

RESEARCH STATEMENT

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1. INTRODUCTION

Over the last twenty years, there has been a great deal of interaction between mathematics and physics, to the mutual advantage of both. In particular, I would like to focus on how ideas inspired by physics have contributed to several new ‘discoveries’ in algebraic geometry in the recent decades, and how conjectures from algebraic geometry have contributed to the development of new physical theories. The physical theories I am referring to are closely related to several moduli spaces that naturally appear in algebraic geometry.

For example, Yang-Mills gauge theories play an important role both in physics, where they describe the physics of particle interactions, and in mathematics, where they are really important in the studying of the moduli spaces of vector bundles on a curve. In the late ’80s and early ’90s, a great deal of thought went into finding a mathematical proof for the surprising Verlinde formula, which gave a prediction for the dimension of the space of sections of a natural line bundle on these moduli spaces (for a survey on this, see [Bea95]). Besides improving the mathematical understanding of the moduli spaces of vector bundles on a curves, other moduli spaces had to be introduced and studied, like the moduli space of parabolic bundles or the moduli space of stable pairs.

Another example is mirror symmetry, which is of great interest to both physicists and mathematicians (a reference is [MirSym03]). Mirror symmetry started in string theory from the observation that certain worldsheet theories did not uniquely determine the corresponding manifold, but determined a pair of manifolds instead. These two manifolds were called “mirrors” of each other, and several predictions were made on how they relate to each other. Just as for the Verlinde formula, a lot of effort went into finding a mathematical proof of these predictions, except that in this case mathematicians even had to first find a precise mathematical statement. Mirror symmetry gives a relation between certain (relatively easy to calculate) invariants of a manifold and other (hard to calculate) invariants on the “mirror” manifold, and this has been used to solve many enumerative questions in algebraic geometry. An example of these invariants are the Gromov-Witten invariants, and their study led to a better understanding of the moduli space of curves. Moreover, just as in the previous example, new moduli spaces had to be introduced, like the moduli space of stable maps or the moduli space of admissible covers.

Several proofs and new conjectures arose from the predictions of mirror symmetry, and the new conjectures helped in the development of new physical theories. For example, a conjecture by Kontsevich in [Kon95], inspired by mirror symmetry, predicted an equivalence between two categories, the derived category of coherent sheaves on a manifold, and the Fukaya category of the “mirror.” This caught physicists by surprise, and new theories have been developed to find a physical explanation for this conjecture. This is related to the concept of π -stability for objects in the derived category, and after Bridgeland gave a precise

corresponding definition of stability condition in [Bri1], several algebraic geometers started studying moduli spaces of stability conditions.

2. MY RESEARCH

My research has involved several of the moduli spaces mentioned in the introduction, and I have decided to group my results into three sections.

2.1. Moduli Spaces of Stable Objects in the Derived Category. Derived categories have started to play a greater role in algebraic geometry in the past few years, and, as mentioned in the introduction, several people have now been studying moduli spaces of stability conditions as defined by Bridgeland.

In joint work with A. Bertram (see [ArcBer]), we took the next step, and we started studying moduli spaces of stable objects in the derived category, i.e., instead of studying a space classifying stability conditions, we study a space classifying stable objects for a given stability condition.

Our main result is the construction of moduli spaces of stable objects with given fixed invariants for a 1-parameter family of stability conditions on a K3 surface. These moduli spaces generalize the moduli spaces of torsion-free sheaves on S . Indeed, for certain stability conditions, the only stable objects with given fixed invariants are torsion-free sheaves (see [Bri2]).

We prove that the moduli spaces we study exist, and are smooth proper symplectic varieties. Moreover, they are related via Mukai flops. The main ingredient of the proof is the characterization of the stable objects with our fixed invariants for a given stability condition.

2.2. Moduli Spaces of Curves and their Tautological Rings. In studying the moduli spaces of curves, one important construction is the tautological ring, which is a subring of the Chow ring which contains all natural classes arising from the geometry of these moduli spaces. Unlike the Chow ring, the tautological ring behaves nicely, and it has a nice conjectural description (see [Fab99]). A lot of work has been put into finding relations among its known generators. In distinct joint works, with Y.-P. Lee (see [ArcLee1] and [ArcLee2]) and with F. Sato (see [ArcSat]), we have found new relations via techniques that allow to find many relations in a uniform way.

2.2.1. Joint work with Y.-P. Lee. Y.-P. Lee conjectured that there exists a finite algorithm which can calculate all of the relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$. It is a recursive algorithm, in the sense that, for a fixed pair (g, n) , to find tautological relations in $\overline{\mathcal{M}}_{g,n}$, one needs to know all tautological relations in $\overline{\mathcal{M}}_{g',n'}$ with $g' \leq g$, and $n' \leq N(g, g')$, where $N(g, g')$ is a number that only depends on g and g' .

This conjecture was first proved to be true in genus 0 and genus 1 by Givental and Lee, and then in genus 2 for $n \leq 3$ in [ArcLee1], i.e., we used the conjecture to rederive all previously known tautological relation in a uniform way. The important part of this first result is that we were able to produce all of these relations with the same method, while their original individual proofs were very different.

We then used the conjecture to find a relation for ψ^3 in $\overline{\mathcal{M}}_{3,1}$ in [ArcLee2].

2.2.2. *Joint work with F. Sato.* Mumford proved in [Mum83] that $\psi^g - \lambda_1\psi^{g-1} + \dots + (-1)^g\lambda_g = 0$ in the Chow ring of the moduli space of smooth curves $\mathcal{M}_{g,1}$. Sato and I found an explicit recursive formula for $\psi^g - \lambda_1\psi^{g-1} + \dots + (-1)^g\lambda_g$ in the tautological ring of $\overline{\mathcal{M}}_{g,1}$ as a combination of boundary strata (see [ArcSat]).

The nice thing about this formula is that it is recursive. The classes in the formula supported on the boundary are a product of two classes on lower genus curves, one of which can be calculated via the the same formula. We explicitly calculated the answer for $g \leq 5$ ¹.

2.3. Moduli Spaces of Torsion-free Shaves on Irreducible Curves. These moduli spaces were the object of study of my Ph.D. Thesis. I worked both on extending some known results to the case of nodal curves (see [Arc]) and on proving new results in the case of smooth curves, trying to answer open questions raised in [Bea95] (see [Arc05]).

2.3.1. *Extension Spaces and the Moduli Space of Rank 2 Vector Bundles with fixed Determinant.* For smooth curves, Bertram used the extension spaces $\text{Ext}_C^1(L, \mathcal{O}_C)$ to study $\mathcal{S}U_C(2, L)$, the moduli space of rank 2 vector bundles with determinant L (see [Ber89], [Ber92]).

In my Ph.D. Thesis and [Arc], I extended his results to an irreducible projective curve C with only nodes as singularity, and I used it to study the compactification $\overline{\mathcal{S}U_C(2, L)}$ of $\mathcal{S}U_C(2, L)$ defined using torsion-free sheaves (see [New78] and [Ses82]).

Among the several applications, I proved that the results for a smooth curve of genus 1 (see [Ati57] and [Tu93]) generalize to the irreducible nodal case, and that using a line bundle L of high enough degree, we can prove that $A_{3g-4}(\overline{\mathcal{S}U_C(2, L)}) \simeq \mathbb{Z}$, generalizing the same statement for $A_{3g-4}(\mathcal{S}U_C(2, L))$ proved by Bhosle (see [Bho99] and [Bho04]).

2.3.2. *The Base Locus of the Linear Systems of Generalized Theta Divisors.* The moduli space $\mathcal{S}U_C(r, \mathcal{O}_C)$ of rank r vector bundles with trivial determinant on a smooth curve has a cyclic Picard group (see [DreNar89]). A divisor corresponding to the ample generator of the Picard group is called a “generalized theta divisor,” and the spaces of sections of its multiples are related to the Verlinde formula.

There are many open questions in this area (for a survey, see [Bea95] and the more recent [Bea]). In particular, several people have found vector bundles in the base locus of the linear system of the generalized theta divisor (see [Ray82], [Pop99], [Sch]) In my paper [Arc05], I proved a result that allows to improve the results of their papers, and I found a lower bound for the dimension of the base locus of the generalized theta divisor.

3. FUTURE DEVELOPMENTS

3.1. Moduli Spaces of Stable Objects in the Derived Category. The original purpose of my joint work with A. Bertram was to study Thaddeus-type flips (see [Tha94]) for a surface embedded in projective space. We believe that the Mukai flops we constructed restrict to Thaddeus-type flips on our K3 surface.

Other future topics of research are to use the same technique to construct other moduli spaces for different invariants, and then to extend these results to other surfaces.

¹The answer was already known for $g = 1$ and $g = 2$, but unknown for higher g 's.

3.2. Moduli Spaces of Curves and their Tautological Rings.

3.2.1. *Joint work with Y.-P. Lee.* I am currently working with an undergraduate student on writing a computer program which would implement the algorithm of Y.-P. Lee's conjecture. If successful, we should be able to use the computer program to find new relations. We also would like to find the Betti numbers for the moduli spaces $\overline{\mathcal{M}}_{g,n}$ for low g 's and n 's.

3.2.2. *Joint work with F. Sato.* We believe that the formula we found for $\psi^g - \lambda_1\psi^{g-1} + \dots + (-1)^g\lambda_g$ is just the first step of an algorithm that should calculate each of the classes $\psi^g, \lambda_1\psi^{g-1}, \dots, \lambda_{g-1}\psi, \lambda_g$ in terms of boundary strata. Graber and Vakil proved in [GraVak] that the classes $\psi^g, \lambda_1\psi^{g-1}, \dots, \lambda_{g-1}\psi, \lambda_g$ are supported on the boundary strata with at least one genus 0 component. Our algorithm, if successful, would complement their result by producing an explicit formula. So far, we proved this for $g \leq 3$.

Note: In each section, I tried to give the main idea of the results without going into too many details to keep this statement to a reasonable length. A more detailed version of my research statement is available on my web page at <http://www.math.utah.edu/~arcara/pro/>

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