STABILITY CONDITIONS ON DERIVED CATEGORIES

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This are notes written in the attempt of understanding Bridgeland's papers [Bri02] and [Bri03].

1. Stability of Vector Bundles on Curves

The motivation for the definition of stability comes from the basic definition of stability for a vector bundle on a smooth projective curve C. It is based on two fundamental orderings: the partial ordering arising from the notion of a sub-bundle, and the numerical ordering coming from the slope-function $\mu(E) = \deg(E)/\mathrm{rk}(E)$.

Definition. A vector bundle E on C is called **semi-stable** if $\mu(F) \leq \mu(E)$ for every non-zero sub-bundle $F \subseteq E$. Or, equivalently, if $\mu(G) \geq \mu(E)$ for every non-zero quotient G of E.

There are two basic facts about semi-stable vector bundles.

Remarks. (1) If E_1 and E_2 are semi-stable, and $\mu(E_1) > \mu(E_2)$, then $\mathrm{Hom}\,(E_1, E_2) = 0$.

(2) For every vector bundle E there exists a unique filtration

$$0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = E,$$

called the **Harder-Narasimhan filtration**, with E_j/E_{j-1} semi-stable of slope μ_j for every j, and $\mu_1 > \mu_2 > \cdots > \mu_n$. Clearly, E is semi-stable if and only if $n = 1$.

Idea of the Proof. (1) Let $f\colon E_1 \to E_2$ be a morphism, and let $F \subseteq E_2$ be its image. Since E_2 is semi-stable, if $F \neq 0$, then $\mu(F) \leq \mu(E_2)$. But E_1 is semi-stable and F is a quotient of E_1, and therefore $\mu(E_1) \leq \mu(F)$, a contradiction unless $F = 0$.

(2) If E is already semi-stable the filtration is obviously $0 = E_0 \subseteq E_1 = E$. If E is not semi-stable, let E_1 be the sub-bundle of maximal rank among all the sub-bundles of maximal slope. Then E_1 is semi-stable because it has maximal slope, and we can repeat the construction using the quotient. To repeat the process, it suffices to prove that $\mu(F) < \mu(E_1)$ for all non-zero sub-bundles F of E/E_1. Let F be a sub-bundle of E/E_1. Then there exists $G \subseteq E$ such that

$F = G/E_1$. Since E_1 has maximal slope, $\mu(G) \le \mu(E_1)$; moreover, since E_1 has maximal rank among the sub-bundles with slope $\mu(E_1)$, $\mu(G) < \mu(E_1)$. Then,

$$\mu(F) = \frac{r(G)\mu(G) - r(E_1)\mu(E_1)}{r(G) - r(E_1)} < \frac{r(G)\mu(E_1) - r(E_1)\mu(E_1)}{r(G) - r(E_1)} = \mu(E_1).$$

We can then repeat the process until we obtain a quotient E/E_{n-1} which is semi-stable. □

2. Some Stability Conditions on Abelian Categories

To generalize the notion of stability to an abelian category, we need to define a slope-function.

Definition. The **Grothendieck group** $K(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the quotient of the free abelian group generated by the objects of $\mathcal{A}$ modulo relations $E = F + G$ for every short exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}$.

Definition. A **centered slope-function** on $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that $Z(E)$ lies in the strict upper half-plane $H = \{r \exp(i\pi\phi) \mid r > 0 \text{ and } 0 < \phi \le 1\}$ for all non-zero $E \in \mathcal{A}$. Given a slope-function, the **phase** of a non-zero object $E \in \mathcal{A}$ is defined to be

$$\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1].$$

The phase plays the role that the slope played for vector bundles on a curve.

Definition. Let $Z\colon Z(\mathcal{A}) \to \mathbb{C}$ be a centered slope-function on an abelian category $\mathcal{A}$. A non-zero object $E \in \mathcal{A}$ is called **semi-stable** (with respect to Z) if $\phi(F) \le \phi(E)$ for every non-zero sub-object $F \subseteq E$. Or, equivalently, if $\phi(G) \ge \phi(E)$ for every non-zero quotient G of E.

Example. Let C be a smooth projective curve, and let $\mathcal{A} = \mathcal{M}_{coh}(\mathcal{O}_C)$. Consider the centered slope-function

$$Z(E) = -\deg(E) + i\,\mathrm{rk}(E)$$

on $\mathcal{A}$. An object $E \in \mathcal{A}$ is semi-stable with respect to Z if and only if it is semi-stable in the sense of the definition in section 1 (extend that definition to all non-zero coherent sheaves on C with the convention that $\mu(E) = +\infty$ if $\mathrm{rk}(E) = 0$).

Not every centered slope-function gives a stability condition with a Harder-Narasimhan filtration for every non-zero object E of $\mathcal{A}$.

Proposition ([Bri02, 2.4])**.** *Let $Z\colon Z(\mathcal{A}) \to \mathbb{C}$ be a centered slope-function on an abelian category $\mathcal{A}$. If the two following conditions hold, then every non-zero object E of $\mathcal{A}$ has a (unique) Harder-Narasimhan filtration. If this is the case, we say that Z has the* **Harder-Narasimhan property***.*

- *There are no infinite sequences of sub-objects in $\mathcal{A}$*

$$\cdots \subseteq E_{j+1} \subseteq E_j \subseteq \cdots \subseteq E_2 \subseteq E_1$$

  with $\phi(E_{j+1}) > \phi(E_j)$ for all j.
- *There are no infinite sequences of quotients in $\mathcal{A}$*

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots$$

  with $\phi(E_j) > \phi(E_{j+1})$ for all j.

3. Stability Conditions on Derived Categories

Remark. Even if we shall consider the derived category of an abelian category, the definitions and results of this section make sense for any triangulated category.

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ be the bounded derived category.

Definition. The **Grothendieck group** $K(\mathcal{D})$ of $\mathcal{D}$ is the quotient of the free abelian group generated by objects of $\mathcal{D}$ modulo relations $X = Y + Z$ for any distinguished triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow T(Y).$$

Remark. The functor $D : \mathcal{A} \to \mathcal{D}$ induces an isomorphism $K(\mathcal{A}) \to K(\mathcal{D})$. Its inverse is given by $X \mapsto \oplus(-1)^p H^p(X)$.

We are now ready to define a stability condition on $\mathcal{D}$.

Definition. A **stability condition** $(Z, \mathcal{P})$ on the derived category $\mathcal{D}$ consists of a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ called the central charge, and full additive subcategories $\mathcal{P}(\phi) \subsetneq \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

(S1) If $X \in \mathcal{P}(\phi)$, then $Z(X) = m(X)\exp(i\pi\phi)$ for some $m(X) \in \mathbb{R}_{>0}$.
(S2) For all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
(S3) If $X_j \in \mathcal{P}(\phi_j)$ for $j = 1, 2$ and $\phi_1 > \phi_2$, then $\operatorname{Hom}_{\mathcal{D}}(X_1, X_2) = 0$.
(S4) For each non-zero object $X \in \mathcal{D}$ there exists a finite collection of distinguished triangles

$$X_{j-1} \to X_j \to A_j \to X_{j-1}[1] \quad (1 \le j \le n)$$

with $X_0 = 0$, $X_n = X$, and $A_j \in \mathcal{P}(\phi_j)$ for all j, such that $\phi_1 > \phi_2 > \cdots > \phi_n$.

The main result we want to prove about stability conditions is the following.

Proposition 3.1 ([Bri02, 5.3])**.** *To give a stability condition on $\mathcal{D}$ is equivalent to giving a bounded t-structure on $\mathcal{D}$ and a centered slope-function on its core with the Harder-Narasimhan property.*

Before we prove the proposition, let us go through a little review of t-structures.

Definition. A **t-structure** on $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ such that, if we let $\mathcal{D}^{\le n} = T^{-n}(\mathcal{D}^{\le 0})$ and $\mathcal{D}^{\ge n} = T^{-n}(\mathcal{D}^{\ge 0})$ for every $n \in \mathbb{Z}$, then

(t1) $\mathcal{D}^{\le 0} \subseteq \mathcal{D}^{\le 1}$, $\mathcal{D}^{\ge 0} \supseteq \mathcal{D}^{\ge 1}$.
(t2) $\operatorname{Hom}_{\mathcal{D}}(X, Y) = 0$ for every $X \in \mathcal{D}^{\le 0}$ and $Y \in \mathcal{D}^{\ge 1}$.
(t3) For any X in $\mathcal{D}$ there exists a distinguished triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow T(Y)$$

such that $Y \in \mathcal{D}^{\le 0}$ and $Z \in \mathcal{D}^{\ge 1}$.

Definition. The **core** of a t-structure $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ on $\mathcal{D}$ is $\mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0}$.

The **standard t-structure** on $\mathcal{D}$ is the one with core $\mathcal{A}$ defined by

$$\mathcal{D}^{\le 0} = \{X \in \mathcal{D} \mid H^p(X) = 0 \ \forall p > 0\}, \quad \mathcal{D}^{\ge 0} = \{X \in \mathcal{D} \mid H^p(X) = 0 \ \forall p < 0\}.$$

Theorem. *The core of a t-structure is an abelian category.*

We now want to characterize the cores of bounded t-structures.

Definition. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is called **bounded** if

$$\bigcup_{n\in\mathbb{Z}} \mathcal{D}^{\leq n} = \mathcal{D} = \bigcup_{n\in\mathbb{Z}} \mathcal{D}^{\geq n}.$$

Note that the standard t-structure is bounded.

Lemma 3.2. *A full additive subcategory $\mathcal{A}' \subseteq \mathcal{D}$ is the core of a bounded t-structure if and only if the following two conditions hold:*

(C1) *For every E_1, E_2 objects of $\mathcal{A}'$, and for every $k > 0$, $\operatorname{Hom}_{\mathcal{D}}(E_1, E_2[-k]) = 0$.*

(C2) *For every non-zero object $X \in \mathcal{D}$ there exist integers $m \leq n$ and a collection of triangles*

$$X_{j-1} \longrightarrow X_j \longrightarrow E_j \longrightarrow T(X_{j-1}) \quad (m \leq j \leq n)$$

with $X_{m-1} = 0$, $X_n = X$, $E_m \neq 0$, $E_n \neq 0$, and $E_j[j] \in \mathcal{A}'$ for all j.

Remark. The $E_j[j] \in \mathcal{A}'$ play the role of $H^j(X)$. Indeed, if $\mathcal{A}$ is the core of the standard bounded t-structure, then $X_j = \tau_{\leq j}(X)$ and $E_j = H^j(X) \in \mathcal{A}[-j]$, where $\tau_{\leq j}$ is the truncation functor:

$$\tau_{\leq j}(X)^p = \begin{cases} X^p & \text{if } p < j \\ \ker d^p & \text{if } p = j \\ 0 & \text{if } p > j \end{cases}.$$

Before we prove Lemma 3.2, let us prove the following.

Lemma 3.3. *If $\mathcal{A}'$ satisfies the two conditions $(C1)$ and $(C2)$ of Lemma 3.2, then the collection of triangles in $(C2)$ is unique (up to isomorphisms) for each $X \in \mathcal{D}$.*

Proof. Let $X \in \mathcal{D}$, and suppose that there exist two collection of triangles:

- $X_{j-1} \to X_j \to E_j \to T(X_{j-1})$, $m \leq j \leq n$, $X_{m-1} = 0$, $X_n = X$, $E_m \neq 0$, $E_n \neq 0$, and $E_j[j] \in \mathcal{A}'$ for all j.
- $X'_{j-1} \to X'_j \to E'_j \to T(X'_{j-1})$, $m' \leq j \leq n'$, $X'_{m'-1} = 0$, $X_{n'} = X$, $E'_{m'} \neq 0$, $E'_{n'} \neq 0$, and $E'_j[j] \in \mathcal{A}'$ for all j.

First of all, let us show that $n = n'$ and $m = m'$. Suppose that $n < n'$. Then

$$\operatorname{Hom}_{\mathcal{D}}(E_n, E'_{n'}) = \operatorname{Hom}_{\mathcal{D}}(E_n[n], (E'_{n'}[n'])[n-n']) = 0$$

because $E_n[n], E'_{n'}[n'] \in \mathcal{A}'$ and $n - n' < 0$. The distinguished triangle

$$X_{n-1} \longrightarrow X \longrightarrow E_n \longrightarrow T(X_{n-1})$$

and the associated long exact sequence is

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{D}}(X, E'_{n'}) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(X_{n-1}, E'_{n'}) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(E_n[-1], E'_{n'}) \longrightarrow \cdots$$

Let us show that $\operatorname{Hom}_{\mathcal{D}}(X_{n-1}, E'_{n'}) = 0$. Since $\operatorname{Hom}_{\mathcal{D}}(E_j, E'_{n'}) = \operatorname{Hom}_{\mathcal{D}}(E_j[-1], E'_{n'}) = 0$ for all $m \leq j \leq n-1 < n'-1$, we obtained from the long exact sequences associated to the distinguished triangles $X_{j-1} \to X_j \to E_j \to T(X_{j-1})$ with $m \leq j \leq n-1$ that

$$\operatorname{Hom}_{\mathcal{D}}(X, E'_{n'}) = \operatorname{Hom}_{\mathcal{D}}(X_{n-1}, E'_{n'}) = \operatorname{Hom}_{\mathcal{D}}(X_{n-2}, E'_{n'}) = \cdots = \operatorname{Hom}_{\mathcal{D}}(X_{m-1}, E'_{n'}) = 0.$$

Therefore, $\operatorname{Hom}_{\mathcal{D}}(X, E'_{n'}) = 0$, and $E'_{n'}$ must be 0, a contradiction.

Suppose now that $n = n'$ and $m < m'$. Since

$$\operatorname{Hom}_{\mathcal{D}}(E_m, E'_j[-k]) = \operatorname{Hom}_{\mathcal{D}}(E_m[m], (E'_j[j])[m-j-k]) = 0$$

for every $j > m$ and $k \geq 0$, we obtain using the long exact sequences for the distinguished triangles $X_{j-1} \to X_j \to E_j \to T(X_{j-1})$ for $j > m$ that

$$\operatorname{Hom}_{\mathcal{D}}(E_m, X) = \operatorname{Hom}_{\mathcal{D}}(E_m, X_{n-1}) = \cdots = \operatorname{Hom}_{\mathcal{D}}(E_m, X_{m+1}) = \operatorname{Hom}_{\mathcal{D}}(E_m, E_m) \neq 0.$$

But using the distinguished triangles $X'_{j-1} \to X'_j \to E'_j \to T(X'_{j-1})$ for $j \geq m' > m$, we obtain that

$$\operatorname{Hom}_{\mathcal{D}}(E_m, X) = \cdots = \operatorname{Hom}_{\mathcal{D}}(E_m, X'_{m'}) = \operatorname{Hom}_{\mathcal{D}}(E_m, E'_{m'}) = 0,$$

a contradiction.

Therefore $n = n'$ and $m = m'$. Let us now show that the triangles are unique (up to isomorphisms). Consider the diagram of distinguished triangles

$$\begin{array}{ccccccc} X_{n-1} & \longrightarrow & X & \longrightarrow & E_n & \longrightarrow & T(X_{n-1}) \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & & & \\ X'_{n-1} & \longrightarrow & X & \longrightarrow & E'_n & \longrightarrow & T(X'_{n-1}) \end{array} . \tag{1}$$

We claim that it extends to a unique map of distinguished triangles. To prove this, it suffices to show that $\operatorname{Hom}_{\mathcal{D}}(X_{n-1}, E'_n) = \operatorname{Hom}_{\mathcal{D}}(X_{n-1}, E'_n[-1]) = 0$. As above, it is easy to prove that $\operatorname{Hom}_{\mathcal{D}}(E_j, E'_n[-k]) = \operatorname{Hom}_{\mathcal{D}}(E_j[-1], E'_n[-k]) = 0$ for every $j < n-1$ and every $k \geq 0$. Therefore, we obtain from the distinguished triangles $X_{j-1} \to X_j \to E_j \to T(X_{j-1})$ for $j < n-1$ that

$$\operatorname{Hom}_{\mathcal{D}}(X_{n-2}, E'_n[-k]) = \operatorname{Hom}_{\mathcal{D}}(X_{n-3}, E'_n[-k]) = \cdots = \operatorname{Hom}_{\mathcal{D}}(E_m, E'_n[-k]) = 0$$

for $k = 0, 1$. From the distinguished triangle $X_{n-2} \to X_{n-1} \to E_{n-1} \to T(X_{n-2})$ and the fact that $\operatorname{Hom}_{\mathcal{D}}(E_{n-1}, E'_n[-k]) = 0$ for $k = 0, 1$, we can conclude that $\operatorname{Hom}_{\mathcal{D}}(X_{n-1}, E'_n)$ and $\operatorname{Hom}_{\mathcal{D}}(X_{n-1}, E'_n[-1])$ inject into $\operatorname{Hom}_{\mathcal{D}}(X_{n-2}, E'_n) = 0$ and $\operatorname{Hom}_{\mathcal{D}}(X_{n-2}, E'_n[-1]) = 0$, respectively.

Therefore the diagram (1) completes to a map of distinguished triangles

$$\begin{array}{ccccccc} X_{n-1} & \longrightarrow & X & \longrightarrow & E_n & \longrightarrow & T(X_{n-1}) \\ \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle w} & & \Big\downarrow{\scriptstyle T(u)} \\ X'_{n-1} & \longrightarrow & X & \longrightarrow & E'_n & \longrightarrow & T(X'_{n-1}) \end{array}$$

with u and w unique. Similarly, the diagram

$$\begin{array}{ccccccc} X'_{n-1} & \longrightarrow & X & \longrightarrow & E'_n & \longrightarrow & T(X'_{n-1}) \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & & & \\ X_{n-1} & \longrightarrow & X & \longrightarrow & E_n & \longrightarrow & T(X_{n-1}) \end{array}$$

completes to a diagram

$$\begin{array}{ccccccc} X'_{n-1} & \longrightarrow & X & \longrightarrow & E'_n & \longrightarrow & T(X'_{n-1}) \\ \Big\downarrow{\scriptstyle u'} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle w'} & & \Big\downarrow{\scriptstyle T(u')} \\ X_{n-1} & \longrightarrow & X & \longrightarrow & E_n & \longrightarrow & T(X_{n-1}) \end{array}$$

with u' and w' unique. Since the diagram

$$\begin{array}{ccccccc} X_{n-1} & \longrightarrow & X & \longrightarrow & E_n & \longrightarrow & T(X'_{n-1}) \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & & & \\ X_{n-1} & \longrightarrow & X & \longrightarrow & E_n & \longrightarrow & T(X_{n-1}) \end{array}$$

can be completed both with $u' \circ u, \mathrm{id}, w' \circ w$ and $\mathrm{id}, \mathrm{id}, \mathrm{id}$, both $u' \circ u$ and $w' \circ w$ must be id. Similarly, $u \circ u'$ and $w \circ w'$ must be id. □

Proof of Lemma 3.2. Suppose first that $\mathcal{A}'$ satisfies the two conditions in the lemma. We claim that

$$\mathcal{D}^{\leq 0} = \{X \in \mathcal{D} \mid n \leq 0\}, \quad \mathcal{D}^{\geq 0} = \{X \in \mathcal{D} \mid m \geq 0\}$$

defines a bounded t-structure with core $\mathcal{A}'$. We need to verify the three axioms (the fact that this t-structure is bounded is obvious).

(t1) This is obvious.

(t2) Let $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$, i.e., $n(X) \leq 0$, $m(Y) \geq 1$, Let

$$X_{j-1} \longrightarrow X_j \longrightarrow E_j \longrightarrow T(X_{j-1}) \quad (m(X) \leq j \leq n(X)) \tag{2}$$

be the collection of triangles associated to X, and let

$$Y_{k-1} \longrightarrow Y_k \longrightarrow F_k \longrightarrow T(Y_{k-1}) \quad (m(Y) \leq k \leq n(Y)) \tag{3}$$

be the collection of triangles associated to Y. Since

$$\mathrm{Hom}_{\mathcal{D}}(E_j, F_k) = \mathrm{Hom}_{\mathcal{D}}(E_j, F_k[-1]) = 0$$

for every $j \leq n(X) < m(Y) \leq k$, we obtain from the distinguished triangles (3) that

$$\mathrm{Hom}_{\mathcal{D}}(E_j, Y) = \mathrm{Hom}_{\mathcal{D}}(E_j, Y_{n(Y)-1}) = \cdots = \mathrm{Hom}_{\mathcal{D}}(E_j, Y_{m(Y)}) = \mathrm{Hom}_{\mathcal{D}}(E_j, F_{m(Y)}) = 0$$

for every $j \leq n(X)$. Moreover, since $\mathrm{Hom}_{\mathcal{D}}(E_j[-1], F_k) = \mathrm{Hom}_{\mathcal{D}}(E_j[-1], F_k[-1]) = 0$ for every $j < n(X)$, $\mathrm{Hom}_{\mathcal{D}}(E_j[-1], Y) = 0$ if $j < n(X)$. From the distinguished triangles (2) with $j < n(X)$ we then obtain that

$$\mathrm{Hom}_{\mathcal{D}}(X_{n(X)-1}, Y) = \mathrm{Hom}_{\mathcal{D}}(X_{n(X)-2}, Y) = \cdots = \mathrm{Hom}_{\mathcal{D}}(X_{m(X)}, Y) = \mathrm{Hom}_{\mathcal{D}}(E_{m(X)}, Y) = 0,$$

and from the distinguished triangle $X_{n(X)-1} \to X \to E_{n(X)} \to T(X_{n(X)-1})$ we obtain that $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ injects into $\mathrm{Hom}_{\mathcal{D}}(X_{n(X)-1}, Y) = 0$.

(t3) Let $X \in \mathcal{D}$. If $X \in \mathcal{D}^{\leq 0}$ or $X \in \mathcal{D}^{\geq 1}$, then the statement is obvious. Therefore, we can assume that $m \leq 0 < n$, where

$$X_{j-1} \longrightarrow X_j \longrightarrow E_j \longrightarrow T(X_{j-1}) \quad (m \leq j \leq n)$$

is the collection of triangles associated to X. Let $Y = X_0$. Then $Y \in \mathcal{D}^{\leq 0}$ because the collection of triangles associated to Y is

$$X_{j-1} \longrightarrow X_j \longrightarrow E_j \longrightarrow T(X_{j-1}) \quad (m \leq j \leq 0)$$

by Lemma 3.3. We claim that the cone Z of $Y \to X$ is an object of $\mathcal{D}^{\geq 1}$.

For each $1 \leq k \leq n$, the diagram

$$\begin{array}{ccccccc}
Y & \longrightarrow & X_{k-1} & \longrightarrow & Z_{k-1} & \longrightarrow & T(Y) \\
{\scriptstyle \mathrm{id}_Y}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle T(\mathrm{id}_Y)} \\
Y & \longrightarrow & X_k & \longrightarrow & Z_k & \longrightarrow & T(Y) \\
\downarrow & & {\scriptstyle \mathrm{id}_{X_k}}\downarrow & & & & \downarrow \\
X_{k-1} & \longrightarrow & X_k & \longrightarrow & E_k & \longrightarrow & T(X_{k-1})
\end{array}$$

can be completed by the octahedron axiom to a diagram

$$\begin{array}{ccccccc} Y & \longrightarrow & X_{k-1} & \longrightarrow & Z_{k-1} & \longrightarrow & T(Y) \\ {\scriptstyle \mathrm{id}_Y}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle T(\mathrm{id}_Y)} \\ Y & \longrightarrow & X_k & \longrightarrow & Z_k & \longrightarrow & T(Y) \\ \downarrow & & {\scriptstyle \mathrm{id}_{X_k}}\downarrow & & \downarrow & & \downarrow \\ X_{k-1} & \longrightarrow & X_k & \longrightarrow & E_k & \longrightarrow & T(X_{k-1}) \\ \downarrow & & \downarrow & & {\scriptstyle \mathrm{id}_{E_i}}\downarrow & & \downarrow \\ Z_{k-1} & \longrightarrow & Z_k & \longrightarrow & E_k & \longrightarrow & T(Z_{k-1}) \end{array}.$$

In particular, the collection of triangles associated to Z is

$$Z_{k-1} \longrightarrow Z_k \longrightarrow E_k \longrightarrow T(Z_{k-1}) \quad (1 \leq k \leq n)$$

where Z_k is the cone of $Y = X_0 \to X_k$ for $0 \leq k \leq n$. Therefore, $Z \in \mathcal{D}^{\geq 1}$ as claimed.

Suppose now that $\mathcal{A}'$ is the core of a bounded t-structure. We need to verify the two conditions of the lemma.

(C1) If E_1 and E_2 are objects of $\mathcal{A}' = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$, and $k > 0$, then $\mathrm{Hom}_{\mathcal{D}}(E_1, E_2[-k]) = 0$ because $E_1 \in \mathcal{D}^{\leq 0}$ and $E_2[-k] \in \mathcal{D}^{\geq k} \subseteq \mathcal{D}^{\geq 1}$.

(C2) Let X be an non-zero object of $\mathcal{D}$. Since the t-structure is bounded,

$$m := \max\{n \in \mathbb{Z} \mid X \in \mathcal{D}^{\geq n}\}$$

exists. Then $X[m] \in \mathcal{D}^{\geq 0}$. There exists a distinguished triangle

$$Y \longrightarrow X[m] \longrightarrow Z \longrightarrow T(Y) \tag{4}$$

with $Y \in \mathcal{D}^{\leq 0}$ and $Z \in \mathcal{D}^{\geq 1}$. We claim that $Y \in \mathcal{A}'$. To prove this, we need to show that $Y \in \mathcal{D}^{\geq 0}$ or, equivalently, that $Y[-1] \in \mathcal{D}^{\geq 1}$. There exists a distinguished triangle

$$Y' \longrightarrow Y[-1] \longrightarrow Z' \longrightarrow T(Y') \tag{5}$$

with $Y' \in \mathcal{D}^{\leq 0}$ and $Z' \in \mathcal{D}^{\geq 1}$. From the distinguished triangle (4) we obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Y', Z[-2]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Y', Y[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Y', X[m-1]) \longrightarrow \cdots$$

Since $Y' \in \mathcal{D}^{\leq 0}$ while $Z[-2]$ and $X[m-1]$ are in $\mathcal{D}^{\geq 1}$, we obtain that $\mathrm{Hom}_{\mathcal{D}}(Y', Y[-1]) = 0$. From the distinguished triangle (5) we obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Y', Z'[-1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Y', Y') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Y', Y[-1]) \longrightarrow \cdots$$

Since $Y' \in \mathcal{D}^{\leq 0}$ and $Z'[-1] \in \mathcal{D}^{\geq 1}$, $\mathrm{Hom}_{\mathcal{D}}(Y', Y') = 0$, $Y' = 0$, and therefore $Y'[-1]$ is isomorphic to $Z' \in \mathcal{D}^{\geq 1}$.

To summarize, starting with a non-zero object X in $\mathcal{D}$, we constructed a distinguished triangle

$$Y[-m] \longrightarrow X \longrightarrow Z[-m] \longrightarrow Y[1-m]$$

with $(Y[-m])[m] \in \mathcal{A}'$. Let $X_m = E_m := Y[-m]$. For $X[m+1]$, there exists a distinguished triangle

$$Y_1 \longrightarrow X[m+1] \longrightarrow Z_1 \longrightarrow T(Y) \tag{6}$$

with $Y_1 \in \mathcal{D}^{\leq 0}$ and $Z_1 \in \mathcal{D}^{\geq 1}$. Let $X_{m+1} := Y_1[-m-1]$. We claim that there exists a map $X_m \to X_{m+1}$ whose cone E_{m+1} is in $\mathcal{A}'[-m-1]$. From the distinguished triangle (6) we obtain a long exact sequence

$$\to \operatorname{Hom}_{\mathcal{D}}(X_m, Z_1[-m-2]) \to \operatorname{Hom}_{\mathcal{D}}(X_m, X_{m+1}) \to \operatorname{Hom}_{\mathcal{D}}(X_m, X) \to \operatorname{Hom}_{\mathcal{D}}(X_m, Z_1[-m-1]) \to$$

Since $X_m[m] \in \mathcal{D}^{\leq 0}$ while $Z_1[-2]$ and $Z_1[-1]$ are in $\mathcal{D}^{\geq 1}$,

$$\operatorname{Hom}_{\mathcal{D}}(X_m, Z_1[-m-j]) = \operatorname{Hom}_{\mathcal{D}}(X_m[m], Z_1[-j]) = 0 \quad (j = 1, 2).$$

Therefore, $\operatorname{Hom}_{\mathcal{D}}(X_m, X_{m+1}) = \operatorname{Hom}_{\mathcal{D}}(X_m, X)$, and there exists a morphism $X_m \to X_{m+1}$ that composed with the morphism $X_{m+1} \to X$ of the distinguished triangle (6) gives us the morphism $X_m \to X$ of the distinguished triangle (4). The distinguished triangle

$$X_m \longrightarrow X_{m+1} \longrightarrow E_{m+1} \longrightarrow T(X_m)$$

associated to this morphism $X_m \to X_{m+1}$ is such that $E_{m+1}[m+1] \in \mathcal{A}'$. We can then continue by taking a distinguished triangle with $X[m+2]$ and so on. The process ends when $X[m+k]$ is in $\mathcal{D}^{\leq 0}$.

□

Idea of the Proof of Proposition 3.1. If $(Z, \mathcal{P})$ is a stability condition on $\mathcal{D}$, then

$$\mathcal{D}^{\leq 0} = \{X \in \mathcal{D} \mid \phi_1 \leq 0\}, \quad \mathcal{D}^{\geq 0} = \{X \in \mathcal{D} \mid \phi_n > 0\}$$

define a bounded t-structure with core

$$\mathcal{P}((0,1]) := \{X \in \mathcal{D} \mid 1 \geq \phi_1 > \cdots > \phi_n > 0\}.$$

Moreover, the central charge Z defines a centered slope-function Z on $\mathcal{P}((0,1])$ with the Harder-Narasimhan property.

Conversely, if $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a bounded t-structure with core $\mathcal{A}'$, and $Z\colon K(\mathcal{A}') \to \mathbb{C}$ is a centered slope-function with the Harder-Narasimhan property, then we can define a stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ by observing that $K(\mathcal{A}') = K(\mathcal{D})$ and by setting, for each $\phi \in \mathbb{R}$,

$$\mathcal{P}(\phi) := \{X \in \mathcal{A}' \mid X[1 - \lceil \phi \rceil] \text{ is semi-stable of phase } \phi - \lceil \phi \rceil + 1 \text{ in } \mathcal{A}' \text{ with respect to } Z\}.$$

□

Example. Let C be a smooth projective curve, and let $\mathcal{A} = \mathcal{M}_{coh}(\mathcal{O}_C)$. Consider the standard t-structure on $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$, and the centered slope-function

$$Z(E) = -\deg(E) + i \operatorname{rk}(E)$$

on $\mathcal{A}$. It induces a stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ with

$$\mathcal{P}(\phi) := \{X \in \mathcal{A} \mid X[1 - \lceil \phi \rceil] \text{ is semi-stable of phase } \phi - \lceil \phi \rceil + 1 \text{ in } \mathcal{A} \text{ with respect to } Z\}$$

All objects of $\mathcal{P}(\phi)$ are of the form $D(E)[n]$ for some semi-stable object E in $\mathcal{A}$.

4. Constructing t-structures

A very useful method for constructing t-structures on a triangulated category is that of torsion pairs, first introduced in [HapReiSma96].

Definition. A **torsion pair** in an abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ which satisfy the following two conditions:

(TP1) $\operatorname{Hom}_{\mathcal{A}}(T, F) = 0$ for every $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

(TP2) Every object $E \in \mathcal{A}$ fits into a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Lemma ([HapReiSma96, 2.1])**.** *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. If $\mathcal{A}$ is the core of a bounded t-structure on $\mathcal{D}$, then the full subcategory*

$$\mathcal{A}^\sharp = \{X \in \mathcal{D} \mid H^j(X) = 0 \text{ for } j \neq -1, 0,\ H^{-1}(X) \in \mathcal{F},\ H^0(X) \in \mathcal{T}\},$$

where H^j is the cohomology with respect to the given t-structure, is the core of the bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ defined by

$$\mathcal{D}^{\leq 0} = \{X \in \mathcal{D} \mid H^i(X) = 0 \text{ for } i > 0,\ H^0(X) \in \mathcal{T}\},$$

$$\mathcal{D}^{\geq 0} = \{X \in \mathcal{D} \mid H^i(X) = 0 \text{ for } i < -1,\ H^{-1}(X) \in \mathcal{F}\}.$$

Proof. By Lemma 3.2, $\mathcal{A}$ satisfies the two conditions (C1) and (C2). We can rewrite $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ as

$$\mathcal{D}^{\leq 0} = \{X \in \mathcal{D} \mid n \leq 0,\ E_0 \in \mathcal{T}\}, \quad \mathcal{D}^{\geq 0} = \{X \in \mathcal{D} \mid m \geq -1,\ E_{-1}[-1] \in \mathcal{F}\}.$$

Let us now verify that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfies the three axioms of a t-structure.

(t1) It is obvious.

(t2) Let $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$. Then $n(X) \leq 0$ and $m(Y) \geq 0$. Let

$$X_{j-1} \longrightarrow X_j \longrightarrow E_j \longrightarrow T(X_{j-1}) \quad (m(X) \leq j \leq n(X)) \tag{7}$$

be the collection of triangles associated to X, and let

$$Y_{k-1} \longrightarrow Y_k \longrightarrow F_k \longrightarrow T(Y_{k-1}) \quad (m(Y) \leq k \leq n(Y)) \tag{8}$$

be the collection of triangles associated to Y. Since

$$\operatorname{Hom}_{\mathcal{D}}(E_j, F_k) = \operatorname{Hom}_{\mathcal{D}}(E_j, F_k[-1]) = 0$$

for every $j < k$, we obtain from the distinguished triangles (8) that

$$\operatorname{Hom}_{\mathcal{D}}(E_j, Y) = \operatorname{Hom}_{\mathcal{D}}(E_j, Y_{n(Y)-1}) = \cdots = \operatorname{Hom}_{\mathcal{D}}(E_j, Y_{m(Y)}) = \operatorname{Hom}_{\mathcal{D}}(E_j, F_{m(Y)}) = 0$$

for every $j < 0$, and, if $n(X) = 0$,

$$\operatorname{Hom}_{\mathcal{D}}(E_0, Y) = \operatorname{Hom}_{\mathcal{D}}(E_0, Y_{n(Y)-1}) = \cdots = \begin{cases} \operatorname{Hom}_{\mathcal{D}}(E_0, F_0) & \text{if } m(Y) = 0 \\ 0 & \text{otherwise} \end{cases}.$$

If $n(X) = m(Y) = 0$, then $E_0 \in \mathcal{T}$ and $F_0 \in \mathcal{F}$, which implies $\operatorname{Hom}_{\mathcal{D}}(E_0, F_0) = 0$ by (TP1), and therefore $\operatorname{Hom}_{\mathcal{D}}(E_j, Y) = 0$ for every $j \leq n(X)$.

Similarly, $\operatorname{Hom}_{\mathcal{D}}(E_j[-1], Y) = 0$ for every $j < -1$, and therefore we obtain from the distinguished triangles (7) that

$$\operatorname{Hom}_{\mathcal{D}}(X_{n(X)-2}, Y) = \operatorname{Hom}_{\mathcal{D}}(X_{n(X)-3}, Y) = \cdots = \operatorname{Hom}_{\mathcal{D}}(X_{m(X)}, Y) = \operatorname{Hom}_{\mathcal{D}}(E_{m(X)}, Y) = 0.$$

If $n(X) = -1, 0$, note that, since $\operatorname{Hom}_{\mathcal{D}}(E_{-1}, Y) = 0$, the long exact sequence from $X_{n(X)-2} \to X_{n(X)-1} \to E_{-1} \to T(X_{n(X)-2})$ shows that $\operatorname{Hom}_{\mathcal{D}}(X_{n(X)-1}, Y)$ injects into $\operatorname{Hom}_{\mathcal{D}}(X_{n(X)-2}, Y) = 0$, and it is therefore also 0, and since $\operatorname{Hom}_{\mathcal{D}}(E_0, Y) = 0$, the long exact sequence from $X_{n(X)-1} \to X_{\to} E_0 \to T(X_{n(X)-1})$ shows that $\operatorname{Hom}_{\mathcal{D}}(X, Y)$ injects into $\operatorname{Hom}_{\mathcal{D}}(X_{n(X)-1}, Y) = 0$.

(t3) Let $X \in \mathcal{D}$, and let

$$X_{j-1} \longrightarrow X_j \longrightarrow E_j \longrightarrow T(X_{j-1}) \quad (m \leq j \leq n)$$

be the collection of triangles associated to X in the original t-structure. For $E_0 \in \mathcal{A}$, there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow E_0 \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Let Y be the cone of the map $T[-1] \to X_{-1}$ defined by the composition $T[-1] \to E_0[-1] \to X_{-1}$. Then the collection of triangles associated to Y in the original t-structure is

$$X_{j-1} \longrightarrow X_j \longrightarrow E_j \longrightarrow T(X_{j-1}) \quad (m \leq j \leq -1), \quad X_{-1} \longrightarrow Y \longrightarrow T \longrightarrow T(X_{-1}),$$

and therefore $Y \in \mathcal{D}^{\leq 0}$. The diagram

$$\begin{array}{ccccccc} X_{-1} & \longrightarrow & Y & \longrightarrow & T & \longrightarrow & T(X_{-1}) \\ \downarrow & & & & \downarrow & & \downarrow \\ X_{-1} & \longrightarrow & X_0 & \longrightarrow & E_0 & \longrightarrow & T(X_{-1}) \end{array}$$

can be completed with a map $Y \to X_0$, which induces a map $Y \to X$. Let Z be the cone of this map $Y \to X$. We want to show that $Z \in \mathcal{D}^{\geq 1}$.

The next step is to prove that the cone of $Y \to X_0$ is isomorphic to F. I am not able to prove this in general, so I shall assume in what follows that we are using the standard t-structure. Then there exists a commutative diagram on $\mathcal{A}$

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Im} d^{-1} & \longrightarrow & Y_0 & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Im} d^{-1} & \longrightarrow & \ker d^0 & \longrightarrow & E_0 & \longrightarrow & 0, \\ & & & & \downarrow & & \downarrow & & \\ & & & & F & = & F & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where Y_0 is defined by the exactness of the diagram. Define $Y \in \mathcal{D}$ as

$$Y^j = \begin{cases} X^j & \text{if } j < 0 \\ Y_0 & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{cases}.$$

It is clear that $Y \in \mathcal{D}^{\leq 0}$. There exists a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_0 & \longrightarrow & \ker d^0 & \longrightarrow & F & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_0 & \longrightarrow & X^0 & \longrightarrow & Z_0 & \longrightarrow & 0, \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{Im} d^0 & = & \operatorname{Im} d^0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where Z_0 is defined by the exactness of the diagram. Let $Z \in \mathcal{D}$ be

$$
Z^j = \begin{cases} 0 & \text{if } j < 0 \\ Z_0 & \text{if } j = 0 \\ X^j & \text{if } j > 0 \end{cases}.
$$

Since $H^j(Z) = 0$ if $j < 0$ and $H^0(Z) = F \in \mathcal{F}$, $Z \in \mathcal{D}^{\geq 1}$. The short exact sequence of complexes

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

induces a distinguished triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow T(Y)$$

in $\mathcal{D}$ with $Y \in \mathcal{D}^{\leq 0}$ and $Z \in \mathcal{D}^{\geq 1}$.

□

Before we continue in our analysis of stability conditions, here is a summary of other results in [HapReiSma96] related to the Lemma above.

Corollary ([HapReiSma96, 2.2])**.** *Let $\mathcal{A}$ be an abelian category and $(\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{A}$. Then the following hold.*

- (a) $\mathcal{B} = \{X \in \mathcal{D}^b(\mathcal{A}) \mid H^j(X) = 0$ *for* $j \neq -1, 0,\ H^{-1}(X) \in \mathcal{F},\ H^0(X) \in \mathcal{T}\}$ *is an abelian category.*
- (b) *The pair $(\mathcal{F}[1], \mathcal{T})$ of full subcategories of $\mathcal{B}$ is a torsion pair in $\mathcal{B}$.*
- (c) *For every $X, Y \in \mathcal{B}$,* $\operatorname{Hom}_{\mathcal{D}^b(\mathcal{B})}(X, Y[k]) \simeq \operatorname{Hom}_{\mathcal{D}^b(\mathcal{A})}(X, Y[k])$ *for* $k = 0, 1$.

Definition ([HapReiSma96])**.** A torsion class $\mathcal{T}$ in a torsion pair $(\mathcal{T}, \mathcal{F})$ is called a **tilting torsion class** if $\mathcal{T}$ is a **cogenerator** for $\mathcal{A}$, i.e., if for all $X \in \mathcal{A}$ there exists a monomorphism $X \to T$ for some $T \in \mathcal{T}$.

Theorem ([HapReiSma96, 3.3])**.** *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on $\mathcal{A}$ such that $\mathcal{T}$ is a tilting torsion class, and let* $\mathcal{B} = \{X \in \mathcal{D}^b(\mathcal{A}) \mid H^j(X) = 0$ *for* $j \neq -1, 0,\ H^{-1}(X) \in \mathcal{F},\ H^0(X) \in \mathcal{T}\}$*. Then the following holds.*

- (a) *If $\mathcal{B}$ has enough projectives, then there exists a triangle equivalence $\mathcal{D}^b(\mathcal{B}) \to \mathcal{D}^b(\mathcal{A})$ which is the identity functor when restricted to $\mathcal{B}$.*

(b) *If $\mathcal{A}$ has enough injectives, then there exists a triangle equivalence $\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B})$ which is the identity functor when restricted to $\mathcal{B}$.*

5. The Space of Stability Conditions

To classify all the stability conditions on a triangulated category, it is necessary to introduce a technical definition to avoid the stability conditions where all of the semi-stable objects are concentrated in one phase. An example of a such "bad" stability condition would be one with $Z(E) = i\,m(E)$ for every $E \neq 0$ in the heart of a t-structure which is not of finite length.

Definition. A stability condition is **locally-finite** if there exists a real number $\eta > 0$ such that for all $t \in \mathbb{R}$ the quasi-abelian category $\mathcal{P}((t-\eta, t+\eta)) \subseteq \mathcal{D}$ is of finite length.

Let $\mathrm{Stab}\,(\mathcal{D})$ be the set of locally-finite stability conditions on $\mathcal{D}$, and consider the **generalized metric** on the set

$$\mathrm{Slice}\,(\mathcal{D}) = \{\mathcal{P} \mid (Z, \mathcal{P}) \in \mathrm{Stab}\,(\mathcal{D}) \text{ for some } Z\}$$

defined by

$$d(\mathcal{P}, \mathcal{Q}) = \inf\{\varepsilon \in \mathbb{R}_{\geq 0} \mid \mathcal{Q}(\phi) \subseteq \mathcal{P}([\phi - \varepsilon, \phi + \varepsilon]) \text{ for all } \phi \in \mathbb{R}\}.$$

If $K(\mathcal{D})$ has finite rank, then the vector space $(K(\mathcal{D}) \otimes \mathbb{C})^*$ has a standard topology, and the inclusion

$$\mathrm{Stab}\,(\mathcal{D}) \subseteq (K(\mathcal{D}) \otimes \mathbb{C})^* \times \mathrm{Slice}\,(\mathcal{D})$$

induces a topology on $\mathrm{Stab}\,(\mathcal{D})$. In general, however, there is no natural choice of topology on $(K(\mathcal{D}) \otimes \mathbb{C})^*$

Define, for each $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}\,(\mathcal{D})$, a function

$$||\cdot||_\sigma \colon (K(\mathcal{D}) \otimes \mathbb{C})^* \longrightarrow [0, \infty]$$

by sending a linear map $U \colon K(\mathcal{D}) \otimes \mathbb{C} \to \mathbb{C}$ to

$$||U||_\sigma = \sup\left\{ \frac{|U(E)|}{|Z(E)|} \,\middle|\, E \text{ is semi-stable in } \sigma \right\}.$$

Note that $||\cdot||_\sigma$ has all of the properties of a norm on the complex vector space $(K(\mathcal{D}) \otimes \mathbb{C})^*$, except that it may not be finite.

For each real number $\varepsilon \in (0, 1/4)$, define a subset

$$\mathcal{B}_\varepsilon(\sigma) = \{\tau = (W, \mathcal{Q}) \in \mathrm{Stab}\,(\mathcal{D}) \mid ||W - Z||_\sigma < \sin(\pi\varepsilon) \text{ and } d(\mathcal{P}, \mathcal{Q}) < \varepsilon\} \subseteq \mathrm{Stab}\,(\mathcal{D}).$$

Note that the condition $||W - Z||_\sigma < \sin(\pi\varepsilon)$ implies that, if E is semi-stable in σ, then the phase of $W(E)$ differs from the phase of $Z(E)$ by less than ε.

Lemma. *The subsets $B_\varepsilon(\sigma)$ form a basis for a topology on* $\mathrm{Stab}\,(\mathcal{D})$ *as ε varies in $(0, 1/4)$ and σ varies in* $\mathrm{Stab}\,(\mathcal{D})$.

The crucial point of the proof is the following result.

Lemma ([Bri02, 6.2])**.** *If $\tau \in B_\varepsilon(\sigma)$, then there exist constants $k_1, k_2 > 0$ such that*

$$k_1||U||_\sigma < ||U||_\tau < k_2||U||_\sigma$$

for all $U \in (K(\mathcal{D}) \otimes \mathbb{C})^*$.

Theorem ([Bri02, 1.2]). *For each connected component $\Sigma \subseteq \mathrm{Stab}\,(\mathcal{D})$ there exists a linear subspace $V(\Sigma) \subseteq (K(\mathcal{D}) \otimes \mathbb{C})^*$ with a well-defined linear topology such that*

$$\mathcal{Z}\colon \Sigma \longrightarrow V(\Sigma) \quad \mathcal{Z}((Z,\mathcal{P})) = Z$$

is a local homeomorphism.

The key step in the proof is the following theorem.

Theorem ([Bri02, 7.1]). *For every $\sigma = (Z,\mathcal{P}) \in \mathrm{Stab}\,(\mathcal{D})$ there exists $\varepsilon_0 > 0$ such that the following holds. If $0 < \varepsilon < \varepsilon_0$, and $W\colon K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism satisfying*

$$|W(E) - Z(E)| < \sin(\pi\varepsilon)|Z(E)|$$

for every $E \in \mathcal{D}$ semi-stable in σ, then there exists $(W,\mathcal{Q}) \in \mathrm{Stab}\,(\mathcal{D})$ with $d(\mathcal{P},\mathcal{Q}) < \varepsilon$.

We conclude the section with a couple of general results on the sapce of stability conditions.

Proposition ([Bri02, 8.1]). *The topology on $\mathrm{Stab}\,(\mathcal{D})$ defined above is induced by the generalized metric*

$$d(\sigma,\tau) = \sup_{0\neq X\in\mathcal{D}} \left\{ \left|\phi^-_\sigma(X) - \phi^-_\tau(X)\right|,\ \left|\phi^+_\sigma(X) - \phi^+_\tau(X)\right|,\ \left|\log\left(\frac{m_\sigma(X)}{m_\tau(X)}\right)\right| \right\} \in [0,\infty].$$

Lemma ([Bri02, 8.2]). *The generalized metric space $\mathrm{Stab}\,(\mathcal{D})$ carries a right action of the universal covering space $\widetilde{GL}^+(2,\mathbb{R})$ of $GL^+(2,\mathbb{R})$, and a left action by isometries of the group $\mathrm{Aut}(\mathcal{D})$ of exact autoequivalences of $\mathcal{D}$. These two actions commute.*

6. Numerical Stability Conditions

While everything up to now could have been defined on any triangulated category $\mathcal{T}$ instead of on $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ for some abelian category $\mathcal{A}$, what follows only makes sense if $\mathcal{T}$ is

- **linear** over a field k, i.e., the morphisms of $\mathcal{T}$ have a structure of a vector space over k, with respect to which the composition law is bilinear.
- of **finite type**, i.e., for every pair of objects $E, F \in \mathcal{T}$, the vector space $\oplus_i \mathrm{Hom}_{\mathcal{T}}(E, F[i])$ is finite-dimensional.
- **numerically finite**, i.e., the numerical Grothendieck group defined below has finite rank.

Let $\mathcal{A} = \mathcal{M}_{coh}(\mathcal{O}_X)$ be the abelian category of coherent sheaves over a smooth algebraic variety X, and let $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ be its bounded derived category.

Definition. There exists a bilinear form on $K(\mathcal{D})$, known as the **Euler form**, defined by

$$\chi(E,F) = \sum_i (-1)^i \dim_k \mathrm{Hom}_{\mathcal{D}}(E, F[i]),$$

and a free abelian groun $\mathcal{N}(\mathcal{D}) = K(\mathcal{D})/K(\mathcal{D})^\perp$, called the **numerical Grothendieck group** of $\mathcal{D}$.

Definition. A stability condition $(Z,\mathcal{P})$ on $\mathcal{D}$ is said to be **numerical** if the central charge Z factors through the quotient group $\mathcal{N}(\mathcal{D})$.

Let $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ be the subspace of $\mathrm{Stab}\,(\mathcal{D})$ consisting of numerical stability conditions. The following is a corollary of [Bri02, Theorem 1.2].

Corollary ([Bri02, 1.3])**.** *For each connected component* $\Sigma \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ *there exists a subspace* $V(\Sigma) \subseteq (\mathcal{N}(\mathcal{D}) \otimes \mathbb{C})^*$ *and a local homeomorphism*

$$\mathcal{Z}\colon \Sigma \longrightarrow V(\Sigma) \quad \mathcal{Z}((Z,\mathcal{P})) = Z.$$

In particular, Σ *is a finite-dimensional complex manifold.*

Example. Let C be a smooth projective curve, let $\mathcal{A} = \mathcal{M}_{coh}(\mathcal{O}_C)$, and let $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$. Then $\mathcal{N}(\mathcal{D}) \simeq \mathbb{Z} \oplus \mathbb{Z}$, and the quotient map $K(\mathcal{D}) \longrightarrow \mathcal{N}(\mathcal{D})$ is given by $E \mapsto (\mathrm{rk}(E), \deg(E))$. Since there exists a numerical stability condition $(Z, \mathcal{P})$ with

$$Z(E) = -\deg(E) + i\,\mathrm{rk}(E)$$

on $\mathcal{A}$, there exists a local homeomorphism

$$\mathcal{Z}\colon \mathrm{Stab}_{\mathcal{N}}(\mathcal{D}) \longrightarrow (\mathcal{N}(\mathcal{D}) \otimes \mathbb{C})^*$$

whose image is some open subset of the two-dimensional vector space $(\mathcal{N}(\mathcal{D}) \otimes \mathbb{C})^*$.

Theorem ([Bri02, 9.1])**.** *If* X *is a non-singular projective curve of genus one, then*

$$\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}) \simeq \widetilde{GL}^+(2,\mathbb{R}).$$

Theorem ([Bri02, 9.2])**.** *If* X *is a non-singular projective curve of genus* $g > 2$*, then there exists a connected component* $\mathrm{Stab}^0_{\mathcal{N}}(\mathcal{D}) \subseteq \mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ *such that*

$$\mathrm{Stab}^0_{\mathcal{N}}(\mathcal{D}) \simeq \widetilde{GL}^+(2,\mathbb{R}).$$

7. K3 Surfaces

Let S be an algebraic K3 surface over $\mathbb{C}$ [1], let $\mathcal{A} = \mathcal{M}_{coh}(\mathcal{O}_S)$, and let $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$. Define a symmetric bilinear form (the **Mukai bilinear form**) on the cohomology ring

$$H^*(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$$

by

$$((r_1, D_1, s_1), (r_2, D_2, s_2)) = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

The resulting lattice $H^*(S,\mathbb{Z})$ is even and non-degenerate and has signature $(4, 20)$. Let Ω be a non-zero holomorphic two-form on S, and let $H^{2,0}(S) \subseteq H^2(S,\mathbb{C})$ denote the one-dimensional complex subspace spanned by the class of Ω. Then the numerical Grothendieck group of $\mathcal{D}$ is

$$\mathcal{N}(S) = \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z} = H^*(S,\mathbb{Z}) \cap \Omega^\perp \subseteq H^*(S,\mathbb{C}).$$

Definition. The **Mukai vector** of an object $E \in \mathcal{D}(S)$ is the element of $\mathcal{N}(S)$ defined by the formula

$$v(E) = (\mathrm{rk}(E), c_1(E), s(E)) = \mathrm{ch}(E)\sqrt{\mathrm{td}(S)} \in H^*(S,\mathbb{Z}),$$

where $\mathrm{ch}(E)$ is the Chern character of E and $s(E) = \mathrm{ch}_2(E) + \mathrm{rk}(E)$.

The Mukai bilinear form makes $\mathcal{N}(S)$ into an even lattice of signature $(2, \rho)$, where $1 \leq \rho \leq 20$ is the Picard number of S. This form is the negative of the Euler form, i.e., for any $E, F \in \mathcal{D}(S)$,

$$\chi(E,F) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}^i_S(E,F) = -(v(E), v(F)).$$

[1]For the results on sheaves on S mentioned here, see also [Muk87].

Let $\mathrm{Stab}(S)$ be the set of numerical locally-finite stability conditions on $\mathcal{D}$. It is the union of finite-dimensional complex manifolds. Remember that a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ is said to be numerical if the central charge Z takes the form

$$Z(E) = (\pi(\sigma), v(E))$$

for some vector $\pi(\sigma) \in \mathrm{NS}(S) \otimes \mathbb{C}$. There is a natural connected component $\Sigma(S) \subseteq \mathrm{Stab}(S)$ with an open subset $U(S)$ where the stability conditions can be described explicitly.

Fix a pair of $\mathbb{R}$-divisors $\beta, \omega \in \mathrm{NS}(S) \otimes \mathbb{R}$ with ω in the ample cone. We can then look at the group homomorphism

$$Z\colon \mathcal{N}(S) \longrightarrow \mathbb{C}$$

defined by $Z(E) = (\exp(\beta + i\omega), v(E))$. More explicitly, if $v(E) = (r, \Delta, s)$, then

$$Z(E) = \begin{cases} \dfrac{1}{2r}(\Delta^2 - 2rs + r^2\omega^2 - (\Delta - r\beta)^2) + i(\Delta - r\beta)\cdot\omega & \text{if } r \neq 0 \\ (\Delta\cdot\beta - s) + i(\Delta\cdot\omega) & \text{if } r = 0 \end{cases}.$$

Now, for a torsion-free sheaf E on S, define

$$\mu_\omega(E) := \frac{c_1(E)\cdot\omega}{\mathrm{rk}(E)},$$

let $\mathcal{F}$ be the set of torsion-free sheaves on S all of whose μ_ω-semi-stable Harder-Narasimhan factors have slope $\mu_\omega \leq \beta\cdot\omega$, let $\mathcal{T}$ be the set of sheaves whose torsion-free parts have μ_ω-semi-stable Harder-Narasimhan factors of slope $\mu_\omega > \beta\cdot\omega$, and let

$$\mathcal{A}(\beta,\omega) := \{E \in \mathcal{D}(S) \mid H^i(E) = 0 \text{ for } i \notin \{-1, 0\},\ H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T}\}.$$

Then $\mathcal{A}(\beta,\omega)$ is the heart of a bounded t-structure on $\mathcal{D}(S)$.

Lemma ([Bri03, 5.2])**.** *Fix a pair of $\mathbb{R}$-divisors $\beta, \omega \in \mathrm{NS}(S) \otimes \mathbb{R}$ with ω in the ample cone. Then the group homomorphism Z defined above is a slope-function on the abelian category $\mathcal{A}(\beta,\omega)$ if and only if $Z(E[1]) \notin \mathbb{R}_{\leq 0}$ for every non-zero torsion-free sheaf E on S which is μ_ω-semi-stable of slope $\mu_\omega(E) = \beta\cdot\omega$.*

Proof. The condition for Z to be a slope-function on $\mathcal{A}(\beta,\omega)$ is that $Z(E)$ is in the strict upper half-plane $H = \{r\exp(i\pi\phi) \mid r > 0 \text{ and } 0 < \phi \leq 1\}$ for all non-zero $E \in \mathcal{A}(\beta,\omega)$. If $E\colon A \to B$ is an element of $\mathcal{A}(\beta,\omega)$, then

$$Z(E) = Z(B) - Z(A) = Z(H^0(E)) - Z(H^{-1}(E)) = Z(H^0(E)) + Z(H^{-1}(E)[1]),$$

and it therefore suffices to show that $Z(E) \in H$ for every $E \in \mathcal{T}$ and $Z(E[1]) \in H$ for every $E \in \mathcal{F}$.

- If $E \in \mathcal{T}$ is supported in dimension 0, then $r = 0$, $\Delta = 0$, and $s = c_2(E) \geq 1$. Therefore $Z(E) \in \mathbb{R}_{<0} \subseteq H$.
- If $E \in \mathcal{T}$ is supported in dimension 1, then $r = 0$ and $\Delta\cdot\omega > 0$. Therefore $\mathrm{Im}\,(Z(E)) > 0$ and $Z(E) \in H$.
- If $E \in \mathcal{T}$ is a torsion-free sheaf, then $\mu_\omega(E) > \beta\cdot\omega$, and $Z(E) \in H$.
- If $E \in \mathcal{F}$ is such that $\mu_\omega(E) < \beta\cdot\omega$, then $Z(E[1]) = -Z(E) \in H$.
- If $E \in \mathcal{F}$ is such that $\mu_\omega(E) = \beta\cdot\omega$, then $Z(E[1]) \in H$ by assumption.

Remark. Note that, if $E \in \mathcal{F}$ is such that $\mu_\omega(E) = \beta\cdot\omega$, then $(\Delta - r\beta)\cdot\omega = 0$, and the Hodge index theorem gives $(\Delta - r\beta)^2 \leq 0$. Moreover, $\Delta^2 - 2rs \geq -2$ by [Muk87, 2.5], and therefore, if $\omega^2 > 2$, then $Z(E[1]) \in H$ for every $E \in \mathcal{F}$.

It can be proved that, when Z is a slope-function on $\mathcal{A}(\beta, \omega)$, it also has the Harder-Narasihman property, and it induces a stability condition in $\mathrm{Stab}(S)$. In all such stability conditions, the skyscreaper sheaves $\mathcal{O}_p$ (with p a point in S) are semi-stable of phase 1. The following proposition gives some kind of converse of this result.

Proposition ([Bri03, 6.2]). *Suppose $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(S)$ is a stability condition on S such that for each point $p \in S$ the sheaf $\mathcal{O}_p$ is stable in σ of phase one. Then there exists a pair $\beta, \omega \in \mathrm{NS}(S) \otimes \mathbb{R}$ with ω in the ample cone such that the heart $\mathcal{A} = \mathcal{P}((0,1])$ of σ is $\mathcal{A}(\beta, \omega)$.*

There exists a connected component $\Sigma(S) \subseteq \mathrm{Stab}(S)$ that contains all of the stability condition constructed above, and a natural open subset $U(S) \subseteq \Sigma(S)$ containing all of the conditions for which $\mathcal{O}_p$ is stable of phase 1 for every point $p \in S$.

We shall now study these stability conditions more in depth.

Definition. A sheaf E on S is called (β, ω)**-twisted semi-stable** if for every sub-sheaf $F \subseteq E$ either

- $\mu_\omega(F) < \mu_\omega(E)$ or
- $\mu_\omega(F) = \mu_\omega(E)$ and $\mathrm{rk}(E) \cdot (s(F) - c_1(F) \cdot \beta) \leq \mathrm{rk}(F) \cdot (s(E) - c_1(E) \cdot \beta)$.

Note that every (β, ω)-twisted semi-stable sheaf is also μ_ω-semi-stable, and that being a $(0, \omega)$-twisted semi-stable sheaf is exactly being a **Gieseker semi-stable** sheaf.

Proposition ([Bri03, 12.2]). *Fix a pair $\beta, \omega \in \mathrm{NS}(S) \otimes \mathbb{Q}$ with ω ample. For integers $n >> 0$, there is a unique stability condition $\sigma_n \in U(S)$ satisfying $\pi(\sigma_n) = \exp(\beta + in\omega)$. Suppose $E \in \mathcal{D}(S)$ satisfies*

$$\mathrm{rk}(E) > 0 \quad \textit{and} \quad (c_1(E) - \mathrm{rk}(E)\beta) \cdot \omega > 0.$$

Then E is semi-stable in σ_n for all $n >> 0$ precisely if E is a shift of a (β, ω)-twisted semi-stable sheaf on S.

Idea of the Proof. The first statement follows as soon as $(n\omega)^2 > 2$ (non-trivial). Note that, for every $n >> 0$, the stability conditions σ_n have the same heart $\mathcal{A}(\beta, \omega)$.

To get an idea of how the proof works, let us prove the proposition in the case $\beta = 0$. Then the central charge Z_n of σ_n simplifies to

$$Z_n(E) = \frac{1}{2} r n^2 \omega^2 - s + in\Delta \cdot \omega.$$

Let E be a sheaf. Then it is easy to see that the argument of $Z_n(E)$ satisfies

$$\lim_{n \to \infty} \frac{1}{\pi} \arg(Z_n(E)) = \begin{cases} 0 & \text{if } \dim \mathrm{supp} E = 2 \\ \frac{1}{2} & \text{if } \dim \mathrm{supp} E = 1 \\ 1 & \text{if } \dim \mathrm{supp} E = 0 \end{cases}.$$

Suppose first that $E \in \mathcal{D}$ is semi-stable in σ_n for all $n >> 0$ and that $\mathrm{rk}(E) > 0$, $c_1(E) \cdot \omega > 0$. Applying a shift, we can assume that $E \in \mathcal{A}(0, \omega)$, and there exists a short exact sequence

$$0 \longrightarrow H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E) \longrightarrow 0$$

in $\mathcal{A}(0, \omega)$. Since the phase of E tends to 0 as $n \to \infty$, and E is semi-stable in σ_n for $n >> 0$, $H^{-1}(E)$ must be zero, or otherwise $H^{-1}(E)[1]$ would be an element of $\mathcal{A}(0, \omega)$ contained in E whose phase would tend to 1 as $n \to \infty$. Therefore $E \simeq H^0(E)$ is a sheaf. Similarly, E must be torsion-free, or is torsion sub-sheaf would be an element of $\mathcal{A}(0, \omega)$ contained in E whose phase would tend to 1 as $n \to \infty$.

Let us now show that E is Gieseker semi-stable. Let $F \subseteq E$ be a sub-sheaf of maximal slope μ_ω. We can assume that $c_1(F) \cdot \omega > 0$ because otherwise F clearly does not destabilize E. Then F is in $\mathcal{A}(0, \omega)$, and since E is semi-stable in σ_n for $n >> 0$, the phase of $Z_n(F)$ must be less than the phase of $Z_n(E)$ for $n >> 0$ [2]. Therefore, for $n >> 0$,

$$\frac{c_1(F) \cdot \omega}{\mathrm{rk}(F) n^2 \omega^2 - 2s(F)} \leq \frac{c_1(E) \cdot \omega}{\mathrm{rk}(E) n^2 \omega^2 - 2s(E)}, \tag{9}$$

which implies that $\mu_\omega(F) \leq \mu_\omega(E)$. Moreover, if $\mu_\omega(F) = \mu_\omega(E)$, then $c_1(F) \cdot \omega$ is equal to $(\mathrm{rk}(F)/\mathrm{rk}(E)) \cdot (c_1(E) \cdot \omega)$, and (9) simplifies to

$$-2s(E)\mathrm{rk}(F) \leq -2s(F)\mathrm{rk}(E),$$

and E is therefore Gieseker semi-stable.

Conversely, supposed that E is a Gieseker semi-stable torsion-free sheaf such that $c_1(E) \cdot \omega > 0$. Then $E \in \mathcal{A}(0, \omega)$. Suppose that

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}(0, \omega)$. Then there exists a distinguished triangle

$$F \longrightarrow E \longrightarrow G \longrightarrow F[1]$$

in $\mathcal{D}$ and a long exact sequence of sheaves

$$0 \longrightarrow H^{-1}(F) \longrightarrow 0 \longrightarrow H^{-1}(G) \longrightarrow H^0(F) \longrightarrow E \longrightarrow H^0(G) \longrightarrow 0.$$

In particular, $H^{-1}(F) = 0$ and $F \simeq H^0(F)$ is also a sheaf. There exists an injective map $(F/H^{-1}(G)) \hookrightarrow E$, and since $\mu_\omega(H^{-1}(G)) \leq 0$ and E is Gieseker semi-stable,

$$\mu_\omega(F) \leq \mu_\omega\left(\frac{F}{H^{-1}(G)}\right) \leq \mu_\omega(E).$$

Moreover, if $\mu_\omega(F) = \mu_\omega(E)$, then $H^{-1}(G)$ must be zero, and $s(F)\mathrm{rk}(E) \leq s(E)\mathrm{rk}(F)$. Therefore, the inequality (9) above is satisfied for all $n >> 0$, and E is semi-stable in σ_n. ☐

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[2]Note that, since E is in $\mathcal{A}(0, \omega)$, all of its Harder-Narasihman slopes must satisfy $\mu_\omega > 0$, and therefore the same is true for the quotient E/F, which is therefore also in $\mathcal{A}(0, \omega)$. This proves that the injective map of sheaves $F \hookrightarrow E$ is also injective in $\mathcal{A}(0, \omega)$.