Section 5.5, Normal Approximations to Binomial Distributions

Let x represent a binomial random variable for n trials, with probability of success for each individual trial p. If $np \geq 5$ and $nq = n(1-p) \geq 5$, then x is approximately normally distributed with mean $\mu = np$ and standard deviation $\sigma = \sqrt{npq}$.

Since the binomial distribution is discrete and the normal distribution is continuous, a "continuity correction" helps to get a better estimate, especially with small sample sizes. To carry this out, intervals will be extended by 0.5 in each direction. For example, to calculate that x is between 10 and 12, we will need to use the interval from 9.5 to 12.5 for this approximation.

Examples

1. The owner of a new apartment building must install 25 water heaters. A certain brand is guaranteed for 5 years, but the probability that it will last 10 years is 0.25. What is the approximate probability that 8 or more of the hot water heaters will last at least 10 years?
   Since $n = 25$, $p = 0.25$, and $q = 0.75$, we know that $np = 6.25$ and $nq = 18.75$. Therefore, the requirements to use the normal approximation are met with $\mu = 6.25$, $\sigma = \sqrt{25 \cdot 0.25 \cdot 0.75} = \sqrt{4.6875} = 2.165$. The continuity correction makes us consider at least 7.5 water heaters instead of 8.

$$P(x \geq 7.5) = P\left(z \geq \frac{7.5 - 6.25}{2.165}\right) = P(z \geq 0.58) = 1 - P(z < 0.58) = 1 - 0.7190$$
$$= 0.2810$$

   Note: The actual probability is 0.2735.

2. From many years of observation, a biologist knows that the probability is only 0.65 that any given Arctic tern will survive the migration from its summer nesting area to its winter feeding grounds. A random sample of 500 Arctic terns were banded at their summer nesting area. What is the approximate probability that between 310 and 340 of the banded Arctic terns will survive the migration?
   The conditions to use a normal approximation are met. We want to look at the probability that a normal random variable with mean $\mu = 500 \cdot 0.65 = 325$ and standard deviation $\sigma = \sqrt{500 \cdot 0.65 \cdot 0.35} = 10.6653$ is between 309.5 and 340.5.

$$P(309.5 \leq x \leq 340.5) = P\left(\frac{309.5 - 325}{10.6653} \leq z \leq \frac{340.5 - 325}{10.6653}\right) = P(-1.45 \leq z \leq 1.45)$$
$$= 0.9265 - 0.0735 = 0.8530$$

3. A professor is giving an exam to a class of 200 students. From past semesters, he knows that 60% of students taking this course receive at least a 70% on this exam. What is the probability that at least 130 of his students will receive a 70% on the test?
   The conditions to use a normal approximation are met, so we want to find the probability that $x \geq 129.5$, where x is a normal random variable with mean $\mu = 200 \cdot 0.6 = 120$ and standard deviation $\sigma = \sqrt{200 \cdot 0.6 \cdot 0.4} = \sqrt{48} = 6.9282$.

$$P(x \geq 129.5) = P\left(x \geq \frac{129.5 - 120}{6.9282}\right) = P(x \geq 1.37) = 1 - 0.9147 = 0.0853$$

4. You flip a weighted coin 10 times (it gives tails 30% of the time, and heads 70%). What is the probability that you will receive at least 9 heads?
   The sample size is not large enough ($10 \cdot 0.3 = 3 < 5$), so we need to use the formulas that

we had for binomial distributions in Section 4.2, with $n = 10$, $p = 0.7$ (since our "success" is heads), and $q = 0.3$. Then,

$$P(x \geq 9) = P(x = 9) + P(x = 10) = \frac{10!}{1!\,9!}\,0.7^9 \cdot 0.3^1 + \frac{10!}{0!\,10!}\,0.7^{10} \cdot 0.3^0 = 0.1493$$

Note: There is a table with steps for this process on page 284 of the book.