Section 5.2, Normal Distributions: Finding Probabilities
Section 5.3, Normal Distributions: Finding Values

In the last section, we focused on finding probabilities for the standard normal distribution. Now, we will consider normal distributions with $\mu \neq 0$ and/or $\sigma \neq 1$.

To find probabilities for variables from a normal distribution with mean μ and standard deviation σ, we will first need to find the z-score to convert to standard normal, then we can use the table that we used in the last section to find probabilities.

We will also look at going from a probability to a range of observations that correspond to that probability. For example, we will look at finding quartiles and percentile. The $p^{\textbf{th}}$ **percentile** is a value such that p percent of the observations fall at or below that value.

Examples

1. Find the 7^{th} percentile for a standard normal distribution.
   We want to find x such that $P(z \leq x) = 0.07$. The decimal on the table closest to .07 is .0694 (corresponding to a z-score of -1.48), so the 7^{th} percentile is approximately -1.48.

2. ACT scores are approximately normal with a mean of 21 and a standard deviation of 4.7.

   (a) What is the probability that a student chosen at random receives less than a 26 on the ACT?
   Let x represent this student's ACT score. The z-score for 26 is $z = \frac{26-21}{4.7} = 1.06$ (Note: You should round to 2 decimal places, since the table goes to 2 decimal places.). So, $P(x < 26) = P(z < 1.06) = 0.8554$.

   (b) What is the probability that a student chosen at random receives at least a 24 on the ACT?
   The z-score for 24 is $z = \frac{24-21}{4.7} = 0.64$, so $P(x \geq 24) = P(z \geq 0.64) = 1 - P(z < 0.64) = 1 - 0.7389 = 0.2611$.

   (c) What is the third quartile for the ACT?
   We want to find y such that $P(x < y) = 0.75$. The closest probability to 0.75 on the table is 0.7486, which corresponds to a z-score of 0.67. Now, we want the y-value that corresponds to a z-score of 0.67. We can do this by solving the equation:

   $$\begin{aligned} \frac{y-21}{4.7} &= 0.67 \\ y - 21 &= 3.149 \\ y &= 24.149, \end{aligned}$$

   so the third quartile is approximately 24.149.

   (d) What is the 80^{th} percentile for the ACT?
   We want to find y such that $P(x < y) = 0.8$. The closest probability to 0.8 on the table is 0.7995, which corresponds to a z-score of 0.84. Again, we want the y-value that corresponds to a z-score of 0.84. We can do this by solving the equation:

   $$\begin{aligned} \frac{y-21}{4.7} &= 0.84 \\ y - 21 &= 3.948 \\ y &= 24.948, \end{aligned}$$

   so the third quartile is approximately 24.948.

3. A brand of cassette decks had a deck life that was normally distributed with a mean of 2.3 years and a standard deviation of 0.4 years.

   (a) What is the probability that the cassette deck will break down less than one year after purchase?

   $$P(x < 1) = P\left(z < \frac{1-2.3}{0.4}\right) = P(z < -3.25) = 0.0006$$

   (b) What is the probability that the cassette deck will last for at least 3 years after purchase?

   $$\begin{aligned} P(x \geq 3) = 1 - P(x < 3) = 1 - P\left(z < \frac{3-2.3}{0.4}\right) = 1 - P(z < 1.75) \\ = 1 - 0.9599 = 0.0401 \end{aligned}$$

   (c) What is the 99^{th} percentile for the life of these cassette decks?
   We want y such that $P(x \leq y) = 0.99$. The closest probability on the table is 0.9901, occurring at $z = 2.33$, so

   $$\begin{aligned} \frac{y-2.3}{0.4} &= 2.33 \\ y - 2.3 &= .932 \\ y &= 3.232, \end{aligned}$$

   so the 99^{th} percentile is approximately 3.232 years.

4. A pizza parlor franchise specifies that the average amount of cheese on a large pizza should be 8 ounces and the standard deviation should be 0.5 ounces.

   (a) What is the probability that a pizza chosen at random has less than 7.3 ounces of cheese?

   $$P(x < 7.3) = P\left(z < \frac{7.3-8}{0.5}\right) = P(z < -1.4) = 0.0808$$

   (b) What is the probability that a pizza has more than 8.95 ounces of cheese?

   $$\begin{aligned} P(x > 8.95) = 1 - P(x \leq 8.95) = 1 - P\left(z \leq \frac{8.95-8}{0.5}\right) = 1 - P(z \leq 1.9) \\ = 1 - 0.9713 = 0.0287 \end{aligned}$$

   (c) What is the probability that a pizza contains between 7.6 and 8.3 ounces of cheese?

   $$\begin{aligned} P(7.6 \leq x \leq 8.3) = P\left(\frac{7.6-8}{0.5} \leq z \leq \frac{8.3-8}{0.5}\right) = P(-.8 \leq z \leq 0.6) \\ = 0.7257 - 0.2119 = 0.5138 \end{aligned}$$

   (d) What is the probability that a pizza has exactly 8 ounces of cheese?
   Since this is a continuous distribution, the probability of taking an exact value is zero. (We can see this by noticing that $P(x = 8) = P(8 \leq x \leq 8)$, so when we convert to z-scores and look up probabilities in the table, we'll be subtracting the same number from itself.)

   (e) What is the least amount of cheese that can be on a pizza that will still place in the top 10% of cheesiest pizzas?
   We want y such that $P(x > y) = 0.1$, which is the same a y that satisfies $P(x < y) = .9$.

0.8997 is the closest probability to 0.9 on the table, so $z = 1.28$. Then, finding y, we solve:

$$\begin{aligned} \frac{y-8}{0.5} &= 1.28 \\ y - 8 &= 0.64 \\ y &= 8.64 \end{aligned}$$

(f) Between what two values does the middle 70% of cheese lie?
We first want y such that $P(-y < z < y) = 0.7$, and we will convert to the x value later. By symmetry, we will have 15% of the observations above y, and 15% below $-y$. This means that 85% of the observations are below y, so y satisfies $P(x < y) = 0.85$. 0.8508 is the closest probability on the table to this, which corresponds to a z-score of 1.04. Changing to a normal distribution with mean 8 and standard deviation 0.5, we get that the middle 70% of cheese lies between $8 - 1.04 \cdot 0.5 = 7.48$ and $8 + 1.04 \cdot 0.5 = 8.52$.