Section 5.1, Introduction to Normal Distributions and the Standard Normal Distribution

In this section, we will talk about a special bell-shaped distribution, the **normal distribution** and learn how to find probabilities that a random normal variable falls in a given range.

A **normal distribution** is a continuous probability distribution for a random variable x with the following properties:

- The mean, median, and mode are equal.
- The curve is bell-shaped and symmetric about the mean.
- The total area under the curve equals 1.
- The curve approaches, but never touches, the x-axis as it extends farther from the mean.
- Between $\mu - \sigma$ and $\mu + \sigma$, the graph curves downward. To the left of $\mu - \sigma$ and to the right of $\mu + \sigma$, the graph curves upward. The points at which the curve changes from curving upward to downward are called **points of inflection**.

The graph of a normal distribution is called the **normal curve**. The equation for the curve is:

$$y = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

The **standard normal distribution** is a normal distribution with $\mu = 0$ and $\sigma = 1$.

Any observation x from a normal distribution can be "converted" to data from a standard normal distribution by calculating a z-score:

$$z = \frac{\text{Value} - \text{Mean}}{\text{Standard deviation}} = \frac{x - \mu}{\sigma}$$

Example
Heights of males at a certain university are approximately normal with a mean of 70.9 inches and a standard deviation of 2.9 inches. Find the z-score for a male who is 6 feet tall.
First, we need to convert 6 feet to inches, so we want a z-score for 72 inches.

$$z = \frac{72 - 70.9}{2.9} = 0.3793$$

1 Finding probabilities

For continuous distributions, the probability that a random variable takes an interval of values is the area under the distribution curve over that interval.

For normal distributions, tables are used to calculate probabilities. Table 4 on pages A16 and A17 in the back of your book gives the probability that a standard normal random variable falls below a certain value z. (We will look at other normal distributions in the next section.)

Examples
Let z be a standard normal random variable:

1. Find the probability that z falls below 0.
   By looking up 0 on the table, we see that the probability is 0.5.

2. Find the probability that z falls below 2.74.
   To find this, we look up 2.74 in the table and see that the probability is 0.9969.

3. Find the probability that z falls below -0.93.
   To find this, we look up -0.93 in the table and see that the probability is 0.1762.

4. Find the probability that z is at least 0.62.
   Looking up 0.62 in the table gives us that the probability that z is less than 0.62 is 0.7324, but these are complementary events, so the probability that we want is $1 - 0.7324 = 0.2676$. (Note the value in the table for -0.62 is also 0.2676. This happens due to symmetry.)

5. Find $P(z \geq -2.6)$.
   Note that $P(z \geq -2.6) = 1 - P(z < -2.6) = 1 - 0.0047 = 0.9953$.

6. Find $P(-0.24 \leq z \leq 0.43)$.
   From the table, we know that $P(z \leq 0.43) = 0.6664$ and $P(z \leq -0.24) = 0.4052$, so the area in between the two values is $0.6664 - 0.4052 = 0.2612$.

7. Find $P(z = 1)$.
   We can think of the above probability as $P(1 \leq z \leq 1)$, then use the reasoning from the last part to get that $P(z = 1) = 0$. We can also get this answer with the "area under the curve" definition for probability, which leads us to a rectangle of width zero, which has area zero.

8. Find $P(z \leq -4)$.
   This number isn't on the table, but by following the patterns, we see that $P(z \leq -4) \approx 0$.

9. Find x such that $P(z \leq x) = 0.9222$.
   Now, we want to find a probability on the table, and use that to get the value for x. Finding 0.9222, we see that it corresponds to a z-value of 1.42, so $x = 1.42$.

10. Find the first quartile for a standard normal distribution.
    Here, we want x such that $P(z \leq x) = 0.25$. The closest probability to 0.25 on the table is 0.2514, corresponding to $x = -0.67$, so $Q_1 \approx -0.67$.