Section 4.2, Binomial Distributions

A **binomial experiment** is a probability experiment that satisfies the following conditions:

- The experiment is repeated for a fixed number of trials, with each trial being independent of the other trials.
- There are exactly two possible outcomes for each trial: success (S) and failure (F).
- The probability of success $P(S)$ is the same for each trial.
- The random variable x counts the number of successful trials.

Our notation will include: n, the number of times a trial is repeated; $p = P(S)$, the probability of success in a single trial; $q = P(F) = 1 - p$, the probability of failure in a single trial; and x, the number of success in the n trials.

For example, if we flip a weighted coin 10 times and count the number of heads (let's say that $P(heads) = 0.6$ and $P(tails) = 0.4$) Then we will consider "heads" as a success, and "tails" as a failure, so $n = 10$, $p = 0.6$, $q = 0.4$, and x will be the count of the number of heads (can be any value between 0 and 10).

Another example easily applies to multiple choice tests. If there are 8 questions on a quiz, where each has 5 possible answers. While completing the quiz, you randomly guess an answer for each question. Then, $n = 8$, $p = 1/5 = 0.2$, $q = 1 - 0.2 = 0.8$, and x is the number of questions that you get correct (any number between 0 and 8). Note: If you are guessing on some questions, but not all, this is no longer a binomial experiment, since the probability of getting a correct answer is different for each "trial" (i.e. question). For example, if you narrow one answer down to 2 possible options, then guess, your probability to get it correct is 0.5, but if you have 4 possible options on another question the probability will be 0.25.

The probability of getting exactly x successes in n trials of a binomial experiment is

$$P(x) =_n C_x p^x q^{n-x} = \frac{n!}{(n-x)!x!} p^x q^{n-x}.$$

Furthermore, the mean is $\mu = np$, the variance is $\sigma^2 = npq$, and the standard deviation is $\sigma = \sqrt{npq}$.

Examples

1. When flipping a weighted coin (with the probability of heads being 0.6), what is the probability that it will come up heads exactly 5 times when it is flipped 10 times?

   This describes a binomial distribution with $n = 10$, $p = 0.6$, and $q = 0.4$. We want to find

   $$P(5) = \frac{10!}{5!5!} 0.6^5 \cdot 0.4^5 = 0.2007.$$

2. When randomly guessing on a multiple choice test with 8 questions, where each question has 4 options, what is the probability that you will get at least 7 questions correct? What is the expected number of questions a student will get correct without studying for the exam? What is the standard deviation?

   First note that $n = 8$, $p = 0.25$, and $q = 0.75$. We want to find

   $$P(7 \text{ or } 8) = P(7) + P(8) = \frac{8!}{1!7!} 0.25^7 \cdot 0.75^1 + \frac{8!}{0!8!} 0.25^8 \cdot 0.75^0 = 0.000381.$$

   (Recall that $0! = 1$ and $0.75^0 = 1$.)

The average number of questions a student will get correct is $\mu = 8 \cdot 0.25 = 2$, and the standard deviation is $\sigma = \sqrt{8 \cdot 0.25 \cdot 0.75} = 1.22474$.