Section 3.4, Additional Topics in Probability and Counting

Our goal for this section will be to count the number of ways that an event can happen.

A **permutation** is an ordered arrangement of objects. The number of different permutations of n objects is $n!$. (Recall that $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ for $n \geq 1$ and $0! = 1$.)

Examples

1. How many different ways are there to order 4 objects?
   The number of orderings is $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

2. In how many ways can a standard deck of cards be shuffled?
   Since there are 52 cards in a deck, a standard deck can be shuffled in $52! = 8.0658 \times 10^{67}$.

The number of permutations of n objects taken r at a time is $_nP_r = \frac{n!}{(n-r)!}$ for $r \leq n$.

Examples

1. In how many ways can you arrange 5 cards from a standard deck of cards?
   We can use the above formula with $n = 52$ and $r = 5$, so $_{52}P_5 = \frac{52!}{47!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311,875,200$.

2. Eight runners are racing in a heat for a sprint. In how many ways can they place in first, second, and third?
   We are looking for $_8P_3 = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336$.

The number of distinguishable permutations of n objects, where n_1 are alike, n_2 are alike, ..., and n_k are alike, with $n_1 + n_2 + \cdots + n_k = n$ is $\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$.

Examples

1. In how many ways can you draw 10 balls out of an urn, where 4 of the balls are red, and 6 of the balls are green.
   Since we cannot tell the difference between the 4 red balls, or between the 6 green balls, the number of orders in which we can remove the balls from the urn is $\frac{10!}{4! \cdot 6!} = 210$.

2. A teacher who scales grades plans on assigning 3 A's, 4 B's, 4 C's, 3 D's, and 2 E's to her class of 16 students. How many different ways are there to assign grades?
   Each of the A's are alike, the B's are alike, etc., so the number of way to assign the 16 grades is $\frac{16!}{3! \cdot 4! \cdot 4! \cdot 3! \cdot 2!} = 504,504,000$.

If the order of the selection does not matter, we call the selection a **combination**. The number of combinations of n objects taken r at a time is $_nC_r = \frac{n!}{(n-r)! \cdot r!}$

Examples

1. An NBA team has 15 people on the roster. How many different starting lineups (of 5 people) are there?
   Since order does not matter (the team members are either chosen or not), we are looking for $_{15}C_5 = \frac{15!}{10! \cdot 5!} = 3003$.

2. Jayden needs to choose 7 of his 20 classmates to play on his soccer team. How many different ways are there to choose his team?
   We want to calculate $_{20}C_7 = \frac{20!}{13! \cdot 7!} = 77520$.

3. In how many ways can a 5-card hand in poker be dealt?
   Since the order in which you receive the cards does not matter, we want to calculate $_{52}C_5 = \frac{52!}{47! \cdot 5!} = 2,598,960$.

4. What is the probability of being dealt all hearts in a game of poker with a 5-card hand?

To find the probability, we first need to find the total number of favorable outcomes, which is $_{13}C_5 = 1287$. The total number of outcomes is 2598960, so the probability of being dealt all hearts is $\frac{1287}{2598960} \approx 0.000495$.

5. (Done if we had time) What is the probability of being dealt 4 of a kind in a 5-card hand in poker?

The number of total hands that include four of a kind is $13 \cdot 48 = 624$ (there are 13 different cards of which we can get 4 of a kind, and 48 remaining cards after each "4 of a kind" is selected.), so the probability of getting 4 of a kind is $\frac{624}{2598960} \approx 0.000240$.