

Section 3.2, Conditional Probability and the Multiplication Rule

A **conditional probability** is the probability that an event has occurred, knowing that another event has already occurred. The probability that an event B occurs, given that A has already occurred is denoted $P(B|A)$ and is read “the probability of B given A .”

Example

1. Playing poker with a 5-card hand, you want to know the probability that you get a queen when you replace one card, knowing that you have a queen in your hand, and are trading in a card that is not a queen (which will not be redealt to you).

Since you have a queen, there are only 3 left in the deck. There are only 47 cards that might be dealt to you (you are holding 4, and know that it's not the card you discarded). Therefore, the probability is $3/47$.

2. Using a fair coin, what is the probability of flipping 3 heads in a row, knowing that you have already flipped 2 heads?

Since you only need to flip one more head to achieve the result (which has a probability of $1/2$), this conditional probability is $1/2$.

3. Using the following data, what is the probability that a person you survey has a dog, given that they are male?

Pet Preference					
Gender	Dog	Cat	Other Pet	No Pet	Total
Female	120	132	18	30	300
Male	135	70	20	25	250

135 out of 250 males surveyed had a dog, so the probability that we choose one with a dog is $135/250 = 0.54$.

Two events are called **independent** if the occurrence of one event does not affect the probability of the occurrence of the other event. Two events A and B are independent if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A).$$

Otherwise, we say that the events are **dependent**.

Examples

Determine if each of the following pairs of events is independent or dependent.

1. Selecting an Ace from a deck of cards (Event A), not replacing it, then selecting a King (Event B).

Since $P(B) = 4/52$, and $P(B|A) = 4/51$, the events are dependent.

2. Selecting an Ace from a deck of cards (Event A), replacing it, shuffling the deck, then selecting a King (Event B).

Since $P(B) = 4/52$, and $P(B|A) = 4/52$, the events are independent.

3. Going to college in Utah (Event A), and growing up in Utah (Event B).

These events are dependent, since many students go to college close to home.

4. Today's temperature (Event A), and tomorrow's temperature (Event B).

These events are dependent, since temperatures are normally fairly steady.

5. A person's gender and the choice of their pet, based on the following data:

Pet Preference					
Gender	Dog	Cat	Other Pet	No Pet	Total
Female	120	132	18	30	300
Male	135	70	20	25	250

We already calculated that $P(\text{own a dog} | \text{male}) = 0.54$, so we will now calculate $P(\text{own a dog}) = 255/550 = 0.4536$. Since these numbers are not equal, the variables are dependent. (Note: Some of the difference in proportions can be due to randomness in the sampling, so if two proportions are very close to each other, it is very difficult to tell if events are dependent or not. There is a statistical test (called the Chi-Square test) which gives a more precise answer in these circumstances.)

The probability that two events will occur in sequence is

$$P(A \text{ and } B) = P(A) \cdot P(B | A)$$

If A and B are independent, note that this simplifies to $P(A \text{ and } B) = P(A) \cdot P(B)$.

Examples

1. Two cards are selected from a standard deck without replacement. What is the probability that an Ace is selected, then a King?

$$P(A \text{ and } K) = P(A) \cdot P(K | A) = \frac{4}{52} \cdot \frac{4}{51} = \frac{16}{2652} = \frac{4}{663}$$

2. Calculate the probability that you flip a coin and get a tail, then roll a die and roll either a 2 or 4.

Since these events are independent, $P(T \text{ and } \{2 \text{ or } 4\}) = P(T) \cdot P(2 \text{ or } 4) = \frac{1}{2} \cdot \frac{2}{6} = \frac{1}{6}$

3. Calculate the probability that at least two people in this class (of 25 people) share the same birthday (ignoring leap years).

It is easier to calculate the probability of the complement of the event (that no two people share the same birthday).

$$P(\text{No two students have the same birthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{342}{365} \cdot \frac{341}{365} = 0.4313,$$

so the probability that two students share the same birthday is $1 - 0.4313 = 0.5687$.