Solutions to Sample Problems for Exam 3
Math 1100–4

1. Calculate each of the following integrals:

(a) $\int \left(x^2 + \frac{3}{x^4} - \sqrt[7]{x}\right) dx$

$$\begin{aligned}\int \left(x^2 + \frac{3}{x^4} - \sqrt[7]{x}\right) dx &= \int \left(x^2 + 3x^{-4} - x^{1/7}\right) dx \\ &= \frac{x^3}{3} + 3\frac{x^{-3}}{-3} - \frac{x^{8/7}}{8/7} + C \\ &= \frac{x^3}{3} - x^{-3} - \frac{7x^{8/7}}{8} + C\end{aligned}$$

(b) $\int t\left(4t^2 - 5\right)^7 dt$

$$\begin{aligned}\int t\left(4t^2 - 5\right)^7 dt = \frac{1}{8}\int 8t\left(4t^2 - 5\right)^7 dt &= \frac{1}{8}\frac{\left(4t^2 - 5\right)^8}{8} + C \\ &= \frac{\left(4t^2 - 5\right)^8}{64} + C\end{aligned}$$

(c) $\int \frac{x^3 + 1}{(x^4 + 4x)^8}\, dx$

$$\begin{aligned}\int \frac{x^3 + 1}{(x^4 + 4x)^8}\, dx = \frac{1}{4}\int (4x^3 + 4)(x^4 + 4x)^{-8} &= \frac{1}{4}\frac{(x^4 + 4x)^{-7}}{-7} + C \\ &= -\frac{1}{28}(x^4 + 4x)^{-7} + C\end{aligned}$$

(d) $\int \frac{x^3 + 1}{x^4 + 4x}\, dx$

$$\int \frac{x^3 + 1}{x^4 + 4x}\, dx = \frac{1}{4}\frac{4x^3 + 4}{x^4 + 4x}\, dx = \frac{1}{4}\ln\left|x^4 + 4x\right| + C$$

(e) $\int y^3\, e^{3y^4 - 2}\, dy$

$$\int y^3\, e^{3y^4 - 2}\, dy = \frac{1}{12}\int 12y^3\, e^{3y^4 - 2}\, dy = \frac{1}{12}e^{3y^4 - 2} + C$$

(f) $\int \frac{dx}{x(\ln x)^3}$

$$\int \frac{dx}{x(\ln x)^3} = \int \frac{1}{x}(\ln x)^{-3}\, dx = \frac{(\ln x)^{-2}}{-2} + C = -\frac{1}{2(\ln x)^2} + C$$

(g) $\int e^{\ln z^5}\,dz$

$$\int e^{\ln z^5}\,dz = \int z^5\,dz = \frac{z^6}{6} + C$$

(h) $\int \ln e^{-5x^3/2+6}\,dx$

$$\int \ln e^{-5t^3/2+6}\,dt = \int \left(-\frac{5t^3}{2} + 6\right) dt = -\frac{5t^4}{8} + 6t + C$$

2. The marginal revenue function for a product is $\overline{MR} = \frac{x}{\sqrt{x^2+25}}$. Find the total revenue function.

$$R(x) = \int \overline{MR}\,dx = \int \frac{x}{\sqrt{x^2+25}}\,dx = \frac{1}{2}\int 2x(x^2+25)^{-1/2}\,dx$$
$$= \frac{1}{2}\frac{(x^2+25)^{1/2}}{1/2} + C = (x^2+25)^{1/2} + C$$

To find C, we need to use that $R(0) = 0$:

$$0 = R(0) = (0^2+25)^{1/2} + C = 5 + C$$
$$C = -5$$
$$R(x) = (x^2+25)^{1/2} - 5$$

3. #39–51 odds from Section 12.1.

   Answers in back of book.

4. #43–51 odds from Section 12.2.

   Answers in back of book.

5. #43–51 odds from Section 12.3.

   Answers in back of book.

6. #1–11 odds from Section 12.4.

   Answers in back of book.

7. Calculate each of the following sums:

(a) $\sum_{k=1}^{5} 2k$

$$\sum_{k=1}^{5} 2k = 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 2\cdot 5 = 30$$

or, using our formulas:

$$\sum_{k=1}^{5} 2k = 2\sum_{k=1}^{5} k = 2\frac{5(5+1)}{2} = 30$$

(b) $\displaystyle\sum_{i=1}^{4} i^3$

$$\sum_{i=1}^{4} i^3 = 1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100$$

Note that we do not have a formula for the sum of i^3.

(c) $\displaystyle\sum_{j=2}^{3} (j^2 - 2)$

$$\sum_{j=2}^{3} (j^2 - 2) = (2^2 - 2) + (3^2 - 2) = 9$$

or, using our formulas (note that the formulas start at 1, not 2, so we have to add and subtract the $j = 1$ term):

$$\begin{aligned}\sum_{j=2}^{3} (j^2 - 2) &= \sum_{j=1}^{3} (j^2 - 2) - (1^2 - 2) = \sum_{j=1}^{3} j^2 - \sum_{j=1}^{3} 2 + 1 \\ &= \frac{3(3+1)(2 \cdot 3 + 1)}{6} - 2 \cdot 3 + 1 = 9\end{aligned}$$

8. Approximate the area under $y = x^2 + 3$ from $x = 1$ to $x = 2$ using 4 subintervals and a right-hand approximation.

The width of each subinterval will be $\frac{2-1}{4} = \frac{1}{4}$, so we will be evaluating $y = f(x)$ when $x = \frac{5}{4}$, $x = \frac{3}{2}$, $x = \frac{7}{4}$, and $x = 2$.

$$\begin{aligned}\text{Area} &\approx \frac{1}{4} f\left(\frac{5}{4}\right) + \frac{1}{4} f\left(\frac{3}{2}\right) + \frac{1}{4} f\left(\frac{7}{4}\right) + \frac{1}{4} f(2) \\ &= \frac{1}{4}\left(\frac{25}{16} + 3\right) + \frac{1}{4}\left(\frac{9}{4} + 3\right) + \frac{1}{4}\left(\frac{49}{16} + 3\right) + \frac{1}{4}\left(4 + 3\right) \\ &= \frac{1}{4}\left(\frac{25}{16} + 3 + \frac{9}{4} + 3 + \frac{49}{16} + 3 + 7\right) \\ &= \frac{1}{4}\left(\frac{74}{16} + \frac{9}{4} + 16\right) \\ &= \frac{1}{4}\left(\frac{37}{8} + \frac{9}{4} + 16\right) \\ &= \frac{1}{4}\left(\frac{37}{8} + \frac{18}{8} + \frac{128}{8}\right) = \frac{1}{4}\frac{183}{8} = \frac{183}{32}\end{aligned}$$

9. Calculate the exact area under the curve $y = x^2 + 3$ from $x = 1$ to $x = 2$ using the right-hand approximation.

We can use the formula for the right-hand approximation with $a = 1$, $b = 2$, and $f(x) = x^2 + 3$:

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + i\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right) f\left(1 + i\frac{1}{n}\right)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(\left(1 + i\frac{1}{n}\right)^2 + 3\right)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right)\left(1 + 2i\frac{1}{n} + i^2\frac{1}{n^2} + 3\right)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n} + 2i\frac{1}{n^2} + i^2\frac{1}{n^3}\right)$$
$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{4}{n} + \sum_{i=1}^{n} i\frac{2}{n^2} + \sum_{i=1}^{n} i^2\frac{1}{n^3}\right)$$
$$= \lim_{n\to\infty} \left(\frac{4}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2\right)$$
$$= \lim_{n\to\infty} \left(\frac{4}{n}n + \frac{2}{n^2}\frac{n(n+1)}{2} + \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\right)$$
$$= \lim_{n\to\infty} \left(4 + \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2}\right)$$
$$= 4 + 1 + \frac{1}{3} = \frac{16}{3}$$

(Note: You can double-check this answer by using the "shortcuts" from Section 13.2.)

10. Calculate each of the following definite integrals:

(a) $\displaystyle\int_2^5 \left(x^2 - 3x + 5\right)dx$

$$\int_2^5 \left(x^2 - 3x + 5\right)dx = \left(\frac{x^3}{3} - \frac{3x^2}{2} + 5x\right)\Big|_2^5$$
$$= \frac{125}{3} - \frac{75}{2} + 25 - \left(\frac{8}{3} - \frac{12}{2} + 10\right)$$
$$= \frac{45}{2}$$

(b) $\displaystyle\int_{-2}^1 \frac{dx}{(x-3)^2}$

$$\int_{-2}^1 \frac{dx}{(x-3)^2} = \int_{-2}^1 (x-3)^{-2}\,dx = -(x-3)^{-1}\Big|_{-2}^1$$
$$= -(-2)^{-1} + (-5)^{-1} = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

(c) $\int_0^{\sqrt{\ln 5}} xe^{x^2}\,dx$

$$\int_0^{\sqrt{\ln 5}} xe^{x^2}\,dx = \frac{1}{2}\int_0^{\sqrt{\ln 5}} 2xe^{x^2}\,dx$$
$$= \frac{1}{2}e^{x^2}\Big|_0^{\sqrt{\ln 5}} = \frac{1}{2}e^{\ln 5} - \frac{1}{2}1 = \frac{5}{2} - \frac{1}{2} = 2$$

(d) $\int_3^4 \frac{dx}{x}$

$$\int_3^4 \frac{dx}{x} = \ln|x|\Big|_3^4 = \ln 4 - \ln 3 = \ln\frac{4}{3}$$

(e) $\int_0^\infty xe^{-x^2}\,dx$

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\frac{1}{-2}\int_0^b -2xe^{-x^2}\,dx = \lim_{b\to\infty}\frac{1}{-2}e^{-x^2}\Big|_0^b$$
$$= \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right) = \frac{1}{2}$$

(f) $\int_{-\infty}^{-2} \frac{dx}{x^3}$

$$\int_{-\infty}^{-2} \frac{dx}{x^3} = \lim_{a\to-\infty}\int_a^{-2} x^{-3}\,dx = \lim_{a\to-\infty} -\frac{1}{2}x^{-2}\Big|_a^{-2}$$
$$= \lim_{a\to-\infty}\left(-\frac{1}{2}(-2)^{-2} + \frac{1}{2}a^{-2}\right) = -\frac{1}{8}$$

(g) $\int_2^\infty \frac{dx}{x\ln x}$

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_2^b \frac{\frac{1}{x}}{\ln x}\,dx = \lim_{b\to\infty}\ln\ln x\Big|_2^b = \lim_{b\to\infty}\left(\ln\ln b - \ln\ln 2\right)$$

Since $\ln\ln b \to \infty$ as $b \to \infty$, this integral diverges.

(h) $\int_1^\infty \frac{dx}{\sqrt{x}}$

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b x^{-1/2}\,dx = \lim_{b\to\infty} 2x^{1/2}\Big|_1^b = \lim_{b\to\infty}\left(2b^{1/2} - 1\right)$$

Since $b^{1/2} \to \infty$ as $b \to \infty$, this integral diverges.

11. Calculate the area bounded by the following sets of curves:

(a) $f(x) = 2x^2$, $g(x) = -x - 1$, $x = 1$, and $x = 4$.

Since $f(x) \geq g(x)$ on $[1, 4]$, we can set up the formula for the area as:

$$\begin{aligned}
\text{Area} &= \int_1^4 \big(2x^2 - (-x-1)\big)\, dx = \int_1^4 \big(2x^2 + x + 1\big)\, dx \\
&= \left(\frac{2x^3}{3} + \frac{x^2}{2} + x\right)\bigg|_1^4 = \left(\frac{128}{3} + \frac{16}{2} + 4\right) - \left(\frac{2}{3} + \frac{1}{2} + 1\right) \\
&= \frac{126}{3} + \frac{15}{2} + 3 = 42 + \frac{15}{2} + 3 = \frac{90 + 15}{2} = \frac{105}{2}
\end{aligned}$$

(b) $f(x) = x^2 - 8$ and $g(x) = -x^2$.

We are not given an interval, so the graphs should bound a region. To find the interval, we must first find where the graphs intersect:

$$\begin{aligned}
x^2 - 8 &= -x^2 \\
2x^2 &= 8 \\
x^2 &= 4 \\
x &= \pm 2,
\end{aligned}$$

so the interval we want is $[-2, 2]$. On this interval, $g(x) \geq f(x)$, so the area is:

$$\begin{aligned}
\text{Area} &= \int_{-2}^2 \big(-x^2 - (x^2 - 8)\big)\, dx = \int_{-2}^2 \big(-2x^2 + 8\big)\, dx \\
&= \left(\frac{-2x^3}{3} + 8x\right)\bigg|_{-2}^2 = \left(\frac{-16}{3} + 16\right) - \left(\frac{16}{3} - 16\right) = -\frac{32}{3} + 32 \\
&= \frac{-32 + 96}{3} = \frac{64}{3}
\end{aligned}$$

(c) $f(x) = 16x$ and $g(x) = x^3$.

Again, we aren't given an interval, so we first calculate where the graphs intersect:

$$\begin{aligned}
16x &= x^3 \\
x^3 - 16x &= 0 \\
x(x-4)(x+4) &= 0 \\
x &= 0, \pm 4
\end{aligned}$$

On the interval $[-4, 0]$, $g(x) \geq f(x)$. On the interval $[0, 4]$, $f(x) \geq g(x)$, so the area is:

$$\begin{aligned}
\text{Area} &= \int_{-4}^0 \big(x^3 - 16x\big)\, dx + \int_0^4 \big(16x - x^3\big)\, dx \\
&= \left(\frac{x^4}{4} - 8x^2\right)\bigg|_{-4}^0 + \left(8x^2 - \frac{x^4}{4}\right)\bigg|_0^4 \\
&= 0 - \left(\frac{256}{4} - 128\right) + \left(128 - \frac{256}{4}\right) - 0
\end{aligned}$$

$$= -64 + 128 + 128 - 64 = 128$$

12. Calculate the average value of $f(x) = 6x^2 - 4x + 5$ on the interval $[1, 4]$.

$$\text{Average value} = \frac{1}{4-1}\int_1^4 \left(6x^2 - 4x + 5\right) dx = \frac{1}{3}\left(2x^3 - 2x^2 + 5x\right)\Big|_1^4$$
$$= \frac{1}{3}\Big(\left(2\cdot 64 - 2\cdot 16 + 5\cdot 4\right) - \left(2 - 2 + 5\right)\Big)$$
$$= \frac{1}{3}\left(128 - 32 + 20 - 5\right) = \frac{1}{3}\cdot 111 = \frac{111}{3}$$

13. #1–15 odds from Section 13.4.

Answers in back of book.

14. Given the function $f(x, y) = \ln(xy) + x^2y - 3xy$, find each of the following:

(a) $f(\frac{1}{2}, 2)$

$$f\left(\frac{1}{2}, 2\right) = \ln\left(\frac{1}{2}\cdot 2\right) + \frac{1}{4}\cdot 2 - 3\cdot\frac{1}{2}\cdot 2 = \ln 1 + \frac{1}{2} - 3 = -\frac{5}{2}$$

(b) $f(-e, -e^2)$

$$f(-e, -e^2) = \ln\big(-e\cdot(-e^2)\big) + (-e)^2\cdot(-e^2) - 3(-e)(-e^2)$$
$$= \ln e^3 - e^4 - 3e^3 = 3 - e^4 - 3e^3$$

(c) $f(0, 3)$

This is undefined, since $\ln(0\cdot 3) = \ln 0$ is undefined.

(d) $\frac{\partial}{\partial x} f(x, y)$

$$\frac{\partial}{\partial x} f(x, y) = \frac{1}{x} + 2xy - 3y$$

(e) $\frac{\partial}{\partial y} f(x, y)$

$$\frac{\partial}{\partial y} f(x, y) = \frac{1}{y} + x^2 - 3x$$

(f) $\frac{\partial^2}{\partial x^2} f(x, y)$

$$\frac{\partial^2}{\partial x^2} f(x, y) = -x^{-2} + 2y$$

(g) $\frac{\partial^2}{\partial y^2} f(x, y)$

$$\frac{\partial^2}{\partial y^2} f(x, y) = -y^{-2}$$

(h) $\frac{\partial^2}{\partial x \partial y} f(x, y)$

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = 2x - 3$$

(i) $\frac{\partial^2}{\partial y \partial x} f(x, y)$

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = 2x - 3$$

(j) What is the domain of f?

For the second and third terms, x and y can take any value. Since we can only take the logarithm of positive numbers, we need $xy > 0$, so the domain is all values of x and y that satisfy $xy > 0$.