Solutions for Study Guide for Exam 2
Math 1100-4

1. Find the absolute maxima and minima for $f(x) = x^3 - 2x^2 + x - 3$ on the interval $[-1, 2]$.

   To find absolute extrema on a closed interval, we need to check the critical values, as well as the endpoints of the interval.

$$f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1) = 0$$
$$x = \frac{1}{3}, 1 \text{ Now, we check the value of } f \text{ at the critical values and the endpoints.}$$
$$f(-1) = -7$$
$$f\left(\frac{1}{3}\right) = -\frac{77}{27} \approx -2.852$$
$$f(1) = -3$$
$$f(2) = -1$$

   So, the absolute maximum is -1 and the absolute minimum is -7.

2. The total revenue function for a product is $R(x) = 5600x - 16x^2 - 2x^3$. Find the maximum revenue.

$$R'(x) = 5600 - 32x - 6x^2 = -2(3x+100)(x-28) = 0$$
$$x = -\frac{100}{3}, 28$$

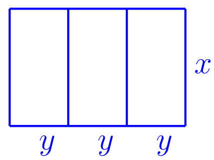
   Since 28 is the only positive critical value, $R(28) = 100352$ is the maximum revenue.

3. The total cost function for a product is $C(x) = 2500 + x^2$. Producing how many units, x, will result in a minimum average cost? Find the minimum average cost.

$$\overline{C} = \frac{2500 + x^2}{x} = 2500x^{-1} + x$$
$$\overline{C}' = -2500x^{-2} + 1 = 0$$
$$1 = \frac{2500}{x^2}$$
$$x^2 = 2500$$
$$x = \pm 50$$

   Since the only positive critical value is 50, producing 50 units will result in a minimum average cost. The minimum average cost is $\frac{2500+50^2}{50} = 100$.

4. Three equal rectangular lots are enclosed by fencing the perimeter of a rectangular lot and then putting two fences across the middle. If each lot is to contain 600 square feet, what is the minimum amount of fencing needed to enclose the lots, including the fences across the middle?



(y represents the width of the smaller rectangular lots.)

We want to minimize the amount of fencing, $F = 4x+6y$. Since this is a function of 2 variables, we need to use $xy = 600$, so $y = \frac{600}{x}$ to eliminate a variable:

$$F = 4x + 6y = 4x + 6 \cdot \frac{600}{x} = 4x + 3600x^{-1}$$
$$F' = 4 - 3600x^{-2} = 0$$
$$4 = 3600x^{-2}$$
$$4x^2 = 3600$$
$$x^2 = 900$$
$$x = 30 \text{ (We can ignore } x = -30\text{, since } x \text{ represents a length.)}$$

Since we were asked for the total amount of fencing, we need to put $x = 30$ into the function that gives the amount of fencing:

$$F = 4x + 3600x^{-1} = 4 \cdot 30 + \frac{3600}{30} = 120 + 120 = 240$$

5. #32 from Section 10.4.

Assign x so that $48 - x$ is the distance the car should be left from the town. Using the equation distance = rate × time, and solving for the time, we know that the time driven will be:

$$\text{time driven} = \frac{\text{distance driven}}{\text{car's speed}} = \frac{48 - x}{55}$$

The time spent on the boat will be calculated in a similar fashion, but we need to find the distance the boat travels using the Pythagorean Theorem ($a^2 + b^2 = c^2$), so

$$8^2 + x^2 = c^2 \text{ so } c = \sqrt{64 + x^2} = (64 + x^2)^{1/2}$$

So,

$$\text{time by boat} = \frac{\text{distance on boat}}{\text{boat's speed}} = \frac{(64 + x^2)^{1/2}}{30}$$

Therefore, the total time T spent in traveling to time is:

$$T = \frac{48}{55} - \frac{x}{55} + \frac{(64 + x^2)^{1/2}}{30}$$

This is the function that we are asked to minimize. Finding the critical values, we see that:

$$\begin{aligned}
T' &= -\frac{1}{55} + \frac{1}{2} \cdot \frac{(64+x^2)^{-1/2}}{30} \cdot 2x = -\frac{1}{55} + \frac{x}{30(64+x^2)^{1/2}} \\
0 &= -\frac{1}{55} + \frac{x}{30(64+x^2)^{1/2}} \\
55x &= 30(64+x^2)^{1/2} \\
\left[11x\right]^2 &= \left[6(64+x^2)^{1/2}\right]^2 \\
121x^2 &= 36(64+x^2) \\
121x^2 &= 2304 + 36x^2 \\
x^2 &= \frac{2304}{85} \\
x &= \sqrt{\frac{2304}{85}} \approx 5.206
\end{aligned}$$

Therefore, the car should be left 5.206 miles from the closest point on the shore to the island (or, 42.794 miles from town).

6. For each of the following functions, find any horizontal and vertical asymptotes, intercept(s), and use information from the first derivative to sketch the graph.

   (a) $f(x) = \dfrac{2x-5}{x+3}$

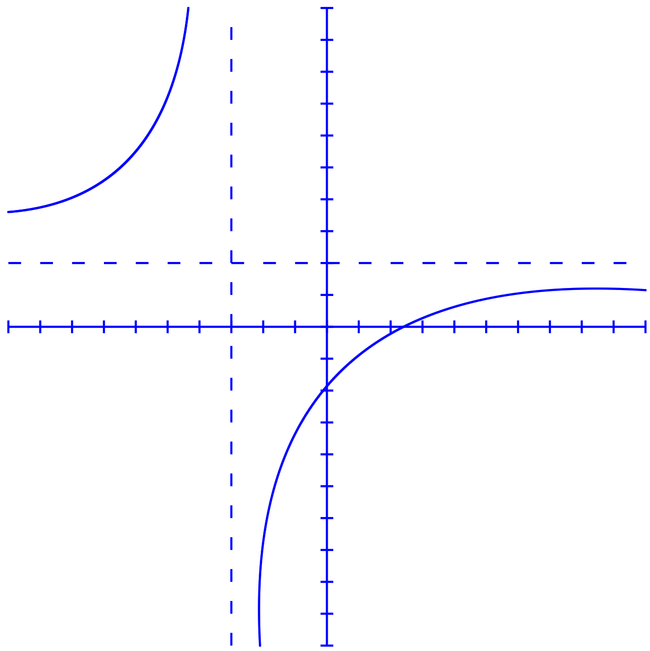
   The vertical asymptote comes from setting the denominator to zero, so it is the line $x = -3$.

   The horizontal asymptote comes from taking the limit of the function as $x \to \infty$, so we get the line $y = 2$.

   The y-intercept (when $x = 0$) is $y = -\frac{5}{3}$. The x-intercept (when $y = 0$) is $x = \frac{5}{2}$.

$$f'(x) = \frac{(x+3)\cdot 2 - (2x-5)\cdot 1}{(x+3)^2} = \frac{11}{(x+3)^2}$$

   This is undefined at $x = -3$ (our vertical asymptote). Otherwise, it is always positive, so $f(x)$ is increasing.

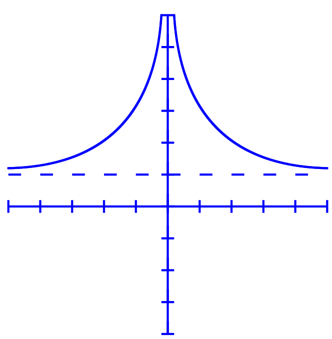


(b) $y = \dfrac{x^2+3}{x^2}$

The vertical asymptote is $x = 0$. The horizontal asymptote is $y = 1$. There are neither x-intercepts (the fraction never equals zero) nor a y-intercept (the function is undefined when $x = 0$.)

$$y' = \frac{x^2(2x) - (x^2+3)2x}{x^4} = -\frac{6}{x^3}$$

y' is undefined at $x = 0$, and is positive for $x < 0$ and negative for $x > 0$, so y is increasing for $x < 0$ and decreasing for $x > 0$.



7. #39 from Section 10.5.

   Answer in back of book.

8. Find the derivative of each of the following functions:

   (a) $f(x) = 3\ln x^2$

   $$f'(x) = 3 \cdot \frac{2x}{x^2} = \frac{6}{x}$$

(b) $y = 3(\ln x)^2$

$$y' = 6(\ln x) \cdot \frac{1}{x} = \frac{6 \ln x}{x}$$

(c) $y = \ln(x^3 - 4)$

$$y' = \frac{3x^2}{x^3 - 4}$$

(d) $f(z) = \dfrac{\ln z^2 + 2z}{4}$

$$f'(z) = \frac{\frac{2z}{z^2} + 2}{4} = \frac{\frac{1}{z} + 1}{2} = \frac{1 + z}{2z}$$

(e) $g(x) = \dfrac{x}{\ln x}$

$$g'(x) = \frac{\ln x - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

(f) $y = \log_6 x^3$

$$y = \frac{\ln x^3}{\ln 6}$$
$$y' = \frac{3x^2}{x^3 \ln 6} = \frac{3}{x \ln 6}$$

(g) $y = e^{2x}$

$$y' = 2e^{2x}$$

(h) $f(x) = 4e^{x^2+1} + x$

$$f'(x) = 4e^{x^2+1} \cdot 2x + 1 = 8xe^{x^2+1} + 1$$

(i) $g(x) = xe^x$

$$g'(x) = xe^x + e^x$$

(j) $h(z) = e^{z^3} + (e^z)^3$

$$h'(z) = 3z^2 e^{z^3} + 3(e^z)^2 e^z = 3z^2 e^{z^3} + 3(e^z)^3$$

(k) $y = 7^{x^2}$

$$y' = 2x \ln 7 \cdot 7^{x^2}$$

(l) $s(t) = \ln(e^t + 1)$

$$s'(t) = \frac{e^t}{e^t + 1}$$

9. #51 from Section 11.2.

   Answer in back of book.

10. Calculate $\frac{dy}{dx}$ for each of the following equations:

   (a) $x^2 - y^2 + 3 = 0$

$$\begin{aligned} 2x - 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= 2x \\ \frac{dy}{dx} &= \frac{x}{y} \end{aligned}$$

   (b) $e^y = x + 2x^2$

$$\begin{aligned} e^y\frac{dy}{dx} &= 1 + 4x \\ \frac{dy}{dx} &= \frac{1 + 4x}{e^y} \end{aligned}$$

   (c) $x^3 - y^2 + y^3 - x^2 + y = 7$

$$\begin{aligned} 3x^2 - 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx} - 2x + \frac{dy}{dx} &= 0 \\ -2y\frac{dy}{dx} + 3y^2\frac{dy}{dx} + \frac{dy}{dx} &= 2x - 3x^2 \\ \frac{dy}{dx}\big(-2y + 3y^2 + 1\big) &= 2x - 3x^2 \\ \frac{dy}{dx} &= \frac{2x - 3x^2}{-2y + 3y^2 + 1} \end{aligned}$$

(d) $(x+y)^2 = 3x^2y$

$$\begin{aligned} 2(x+y)\left(1+\frac{dy}{dx}\right) &= 3x^2\frac{dy}{dx}+6xy \\ 2(x+y)+2(x+y)\frac{dy}{dx} &= 3x^2\frac{dy}{dx}+6xy \\ 2(x+y)\frac{dy}{dx}-3x^2\frac{dy}{dx} &= 6xy-2(x+y) \\ \frac{dy}{dx}\left(2(x+y)-3x^2\right) &= 6xy-2(x+y) \\ \frac{dy}{dx} &= \frac{6xy-2(x+y)}{2(x+y)-3x^2} \end{aligned}$$

11. Find the slope of the line tangent to the graph of $x^2+y^3+x=1$ at the point $(0,1)$.

First, we need to take the derivative using implicit differentiation:

$$\begin{aligned} 2x+3y^2\frac{dy}{dx}+1=0 \\ \frac{dy}{dx} &= -\frac{2x+1}{3y^2}. \text{ Now, we use } x=0 \text{ and } y=1. \\ \frac{dy}{dx} &= -\frac{1}{3} \end{aligned}$$

12. #19 from Section 11.4.

Answer in back of book.

13. #29-35 odds from Section 11.4.

Answers in back of book.

14. #1-9 odds from Section 11.5.

Answers in back of book.