Section 9.8, Higher-Order Derivatives

1 Second Derivatives

If $y = f(x)$ is a function, $f'(x)$ is the **first derivative** of the f. The derivative of the first derivative is called the **second derivative**, and is denoted $f''(x)$, y'', or $\frac{d^2y}{dx^2}$.

Examples

1. Find $\frac{d^2y}{dx^2}$ if $y = 4x^3 - \sqrt[5]{x^4} = 4x^3 - x^{4/5}$.

   To find the second derivative, we need the first derivative, and then we will calculate the derivative of that.

   $$\frac{dy}{dx} = 12x^2 - \frac{4}{5}x^{-1/5}$$
   $$\frac{d^2y}{dx^2} = 24x + \frac{4}{25}x^{-6/5}$$

2. Find $f''(x)$ for $f(x) = 2x^5 - 4x^2 + 2x - 1$.

   $$f'(x) = 10x^4 - 8x + 2$$
   $$f''(x) = 40x^3 - 8$$

3. Calculate $g''(x)$ if $g(x) = 2 - \frac{4}{x^3} = 2 - 4x^{-3}$.

   $$g'(x) = 12x^{-4}$$
   $$g''(x) = -48x^{-5}$$

2 Third Derivatives

The **third derivative** of a function $y = f(x)$ is the derivative of the second derivative, and is denoted $f'''(x)$, y''', or $\frac{d^3y}{dx^3}$.

Examples

1. Let $f(x) = 3x^7 - 9x^5 + 4x^4 - 2$. Calculate $f'''(x)$.

   To find the third derivative, we first need the second derivative.

   $$f'(x) = 21x^6 - 45x^4 + 16x^3$$
   $$f''(x) = 126x^5 - 180x^3 + 48x^2$$
   $$f'''(x) = 630x^4 - 540x^2 + 96x$$

2. $g(x) = \frac{1}{x^3} = x^{-3}$. Find $g'''(x)$.

   $$g'(x) = -3x^{-4}$$
   $$g''(x) = 12x^{-5}$$
   $$g'''(x) = -60x^{-6}$$

Likewise, we can define fourth, fifth, etc. derivatives. Normally, the “prime” notation stops after the third derivative, due to the difficulty in accurately counting large numbers of apostrophes.

3 Application: Marginal Revenue

Example

Find the rate of change of the marginal revenue, $\overline{MR}$, when $x = 20$ if the revenue function is $R(x) = 140x + x^2 - 0.002x^3$ when x units are sold.

Since the $\overline{MR}$ is the first derivative of R, to find the rate at which $\overline{MR}$ is changing, we need the second derivative of R. Then, we will calculate $R''(20)$.

$$\begin{aligned}\overline{MR} = R'(x) &= 140 + 2x - 0.006x^2 \\ R''(x) &= 2 - 0.012x \\ R''(20) &= 2 - 0.012 \cdot 20 = 1.76\end{aligned}$$