

Section 9.5, The Product Rule and the Quotient Rule

1 Product Rule

If $f(x) = u(x) \cdot \nu(x)$, then $f'(x) = u(x) \cdot \nu'(x) + \nu(x) \cdot u'(x)$.

Examples Find the derivative of each of the following functions:

1. $f(x) = x^2(2x^2 + 4x - 1)$

Using our formula, we can let $u(x) = x^2$ and $\nu(x) = 2x^2 + 4x - 1$, so $u'(x) = 2x$ and $\nu'(x) = 4x + 4$, and

$$\begin{aligned} f'(x) &= u(x) \cdot \nu'(x) + \nu(x) \cdot u'(x) = x^2 \cdot (4x + 4) + (2x^2 + 4x - 1) \cdot 2x \\ &= 4x^3 + 4x^2 + 4x^3 + 8x^2 - 2x = 8x^3 + 12x^2 - 2x \end{aligned}$$

2. $g(x) = (x + 2)(x - 3)$

Let $u(x) = x + 2$ and $\nu(x) = x - 3$, so $u'(x) = \nu'(x) = 1$:

$$g'(x) = (x + 2) \cdot 1 + (x - 3) \cdot 1 = 2x - 1$$

3. $h(t) = (\sqrt[3]{t} + t^2)(2t - 4)$

Let $u(t) = \sqrt[3]{t} + t^2 = t^{1/3} + t^2$ and $\nu(t) = 2t - 4$, so $u'(t) = \frac{1}{3}t^{-2/3} + 2t$ and $\nu'(t) = 2$. Then, using the product rule,

$$h'(t) = (\sqrt[3]{t} + t^2) \cdot 2 + (2t - 4) \left(\frac{1}{3}t^{-2/3} + 2t \right)$$

This can be further simplified, if you are asked to do so.

Some notes for the Product Rule:

1. The naming of the functions u and ν is arbitrary, since multiplication is commutative.
2. If one of the functions u or ν is a constant, the product rule simplifies to the Constant Coefficient Rule from the previous section. The product rule can still be used, but it involves more work in this case.
3. For the examples we have looked at so far, it is fairly simple to distribute or multiply out the product, and then take the derivative using the rules from the previous section. Unfortunately, this technique will not work for all the kinds of functions that we will look at this semester, so be sure to practice with the product rule here, before examples become more difficult. (Although, this approach does give you a way to double-check your answers, if you need it.)
4. There are several different sayings to help memorize this rule. One is “the first times the derivative of the second, plus the second times the derivative of the first.” Another that is sometimes used in standard calculus is “ $u \cdot d\nu + \nu \cdot du$,” where “ d ” is short for derivative.

2 Quotient Rule

If $f(x) = \frac{u(x)}{\nu(x)}$, and $\nu(x) \neq 0$, then

$$f'(x) = \frac{\nu(x) \cdot u'(x) - u(x) \cdot \nu'(x)}{[\nu(x)]^2}$$

Examples

Find the derivative of each of the following functions:

1. $f(x) = \frac{x+7}{x-2}$

We need to let $u(x) = x + 7$ and $v(x) = x - 2$, so $u'(x) = v'(x) = 1$. So,

$$f'(x) = \frac{(x-2) \cdot 1 - (x+7) \cdot 1}{(x-2)^2} = \frac{-9}{(x-2)^2}$$

2. $g(t) = \frac{t^2-4t+1}{t-2}$

Here, $u(t) = t^2 - 4t + 1$ and $v(t) = t - 2$. Therefore, $u'(t) = 2t - 4$ and $v'(t) = 1$, so

$$g'(t) = \frac{(t-2)(2t-4) - (t^2-4t+1) \cdot 1}{(t-2)^2} = \frac{2t^2 - 8t + 8 - t^2 + 4t - 1}{(t-2)^2} = \frac{t^2 - 4t + 7}{(t-2)^2}$$

3. $y = \frac{2\sqrt[5]{x}}{x^2+1}$

We start by defining $u(x) = 2\sqrt[5]{x} = 2x^{1/5}$ and $v(x) = x^2 + 1$, so $u'(x) = \frac{2}{5}x^{-4/5}$ and $v'(x) = 2x$.

$$\frac{dy}{dx} = \frac{(x^2+1) \cdot \frac{2}{5}x^{-4/5} - 2\sqrt[5]{x} \cdot 2x}{(x^2+1)^2}$$

For practice, you can simplify this on your own. Some terms in the numerator will combine.

Notes for the Quotient Rule:

1. Unlike in the Product Rule, the naming of u and v is not arbitrary, since division is not commutative.
2. When simplifying, make sure to distribute (especially any negative signs). Also, keeping the denominator factored is fine, since if the fraction reduces at all, you need the numerator and denominator factored. It is also normally easier to evaluate, if needed.
3. If u is constant and v is a power of x , the solution can also be found using the power rule from the previous section (after the fraction is rewritten using exponents).
4. Unlike with the product rule, we normally do not have a simpler way to calculate derivatives of fractional functions. There will be an alternative way to handle derivatives of fractions in the next section, but the calculation will be just as involved in most cases.
5. As with the product rule, there are several different sayings to help memorize this rule. One is “the bottom times the derivative of the top, minus the top times the derivative of the bottom, all over the bottom squared.” Another that is sometimes used in standard calculus is “ $\nu \cdot du - u \cdot d\nu$ over ν squared,” where “ d ” is short for derivative again.

3 Application: Marginal Revenue

The **marginal revenue** is the rate of change of the revenue function. It is sometimes denoted \overline{MR} .

Example

Let $R(x) = 2x^2 - 4x + \frac{2}{x-4}$, where R is the revenue when x units are sold. Answer the following questions:

1. Find the marginal revenue.

We want to calculate $R'(x)$. We can do this using the Sum and Difference Rules, combined with the Power Rule, Constant Coefficient Rule, and the Quotient Rule. (The Product Rule is not needed, as all products involve constants, so we can use the Constant Coefficient Rule instead.)

$$R'(x) = 4x - 4 + \frac{(x-4) \cdot 0 - 2 \cdot 1}{(x-4)^2} = 4x - 4 - \frac{2}{(x-4)^2}$$

2. Find the marginal revenue when 100 units are sold.

We are asked for $R'(100)$, so, using our answer to the previous part, $R'(100) = 400 - 4 - \frac{2}{96^2} \approx 395.9998$.