Section 9.3, Average and Instantaneous Rates of Change: The Derivative

1 Average Rate of Change

The **average rate of change** of a function $y = f(x)$ from $x = a$ to $x = b$ is:

$$\frac{f(b) - f(a)}{b - a}.$$

Note that this equals the slope of the line connecting the points $(a, f(a))$ and $(b, f(b))$.

Example Find the average rate of change of $f(x) = x^2 - 2x + 4$ on the interval $[1, 3]$.

$$\frac{f(3) - f(1)}{3 - 1} = \frac{7 - 3}{2} = 2$$

2 Derivative

If $y = f(x)$ is a function, then the **derivative** of $f(x)$ at any value of x, denoted $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

if this limit exists. If $f'(c)$ exists, we say that f is **differentiable** at c.

If $y = f(x)$, alternative notation for $f'(x)$ includes y', $\frac{dy}{dx}$, $\frac{d}{dx} f(x)$, $D_x y$, and $D_x f(x)$.

The derivative has a variety of interpretations. First, $f'(c)$ is the instantaneous rate of change of the function f at $x = c$. Secondly, $f'(c)$ is the slope of the line tangent to the graph of $y = f(x)$ at $x = c$. We will explore other uses of the derivative later this semester.

Examples

1. Find the instantaneous rate of change of $f(x) = 2x - 4$ at $x = -1$.

   We need to find $f'(-1)$. From the definition of the derivative, with $x = -1$,

   $$f'(-1) = \lim_{h \to 0} \frac{f(-1 + h) - f(-1)}{h} = \lim_{h \to 0} \frac{2(-1 + h) - 4 - (2(-1) - 4)}{h} = \lim_{h \to 0} \frac{2h}{h} = 2$$

2. Find the derivative of $f(x) = x^2 - x + 3$.

   $$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - (x + h) + 3 - (x^2 - x + 3)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x - h + 3 - x^2 + x - 3}{h} = \lim_{h \to 0} \frac{2xh + h^2 - h}{h} \\ &= \lim_{h \to 0} (2x + h - 1) = 2x - 1 \end{aligned}$$

3. Let $f(x) = x^2 - x + 3$ again. Find the slope of the line tangent to the graph of $y = f(x)$ at $x = -3$.

   This question is asking us to find $f'(-3)$. Since we already calculated the derivative, we know that $f'(x) = 2x - 1$, so:

   $$f'(-3) = 2(-3) - 1 = -7$$

We will learn “shortcuts” to calculate derivatives later, but for this section, make sure that you use the definition of the derivative for your calculations.

3 Application: Velocity

The **velocity** of an object is the rate at which its position is changing with respect to time.

Example

The height of a ball thrown upward at a speed of 30 ft/s from a height of 15 feet after t seconds is given by:

$$S(t) = 15 + 30t - 16t^2$$

Find the average velocity of the ball in the first 2 seconds after it is thrown.

We are asked for the average rate of change of the position of the ball between times 0 and 2. Using our formula for average rate of change,

$$\frac{S(2) - S(0)}{2 - 0} = \frac{11 - 15}{2} = -2 \text{ ft/s}$$