Section 9.2, Continuous Functions; Limits at Infinity

1 Continuous Functions

The function f is **continuous** at $x = c$ if all of the following statements are satisfied:

1. $f(c)$ exists.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$.

If one or more of the above conditions does not hold, we say that $f(x)$ is **discontinuous** at $x = c$.

Every polynomial function is continuous for all real numbers.

Every rational function is continuous at all values of x in its domain (i.e. where the denominator is nonzero).

Examples

State where the following functions are discontinuous:

1. $f(x) = 2x^3 + 4x^2 + 3$ is always continuous (polynomial).
2. $g(x) = \frac{3x^2+4x+1}{x-3}$ is discontinuous at $x = 3$ (where the denominator is zero).
3. $h(z) = \frac{2z+1}{(2z+1)(z-2)}$ is discontinuous at $z = -\frac{1}{2}, 2$, since the denominator is zero at those z-values (Do not simplify fractions when finding continuity information. Here, the limit as z approaches $-\frac{1}{2}$ exists, but the function itself is not defined at that point.).
4. $f(x) = \begin{cases} x - 4, & x \leq 4 \\ x^2 + 16, & x > 4 \end{cases}$ is discontinuous at $x = 4$. (For piecewise functions, look for discontinuities within each part of the function, and also be sure to check the x-values where the definition of the function changes.)

If either $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ is infinite, $x = c$ is called a **vertical asymptote** of f.

Examples

1. $f(x) = \frac{x+1}{x-1}$ has a vertical asymptote at $x = 1$.
2. $f(x) = \frac{x^2+4x+4}{x^2-4}$ has a vertical asymptote at $x = 2$. (It does not have a vertical asymptote at $x = -2$, even though the denominator is zero there, since the limit as x approaches -2 exists.)

Note that polynomial functions will not have any vertical asymptotes, but rational functions will have vertical asymptotes whenever the denominator is zero, as long as the numerator is nonzero (if both the numerator and denominator are zero, to check where the vertical asymptotes are, first reduce the fraction completely, then check where the denominator is zero).

2 Limits at Infinity

If $\lim_{x\to\infty} f(x) = b$, or $\lim_{x\to-\infty} f(x) = b$, for a constant b, then the line $y = b$ is a **horizontal asymptote** for the graph of $y = f(x)$. Otherwise, $y = f(x)$ has no horizontal asymptotes.

Properties

1. $\lim_{x\to\infty} c = \lim_{x\to-\infty} c = c$ if c is a constant.

2. $\lim_{x\to\infty} \frac{c}{x^n} = 0$ for any $n > 0$.

3. $\lim_{x\to-\infty} \frac{c}{x^n} = 0$ for $n > 0$, if n is an integer.

For more complicated fractions, you can divide the numerator and denominator of the fraction by the highest power of x found in either polynomial, and then apply these properties.

Examples

1. Find the following limits:

   (a) $\lim_{x\to\infty} \frac{2}{x^4} = 0$, using Property 2.

   (b) $\lim_{x\to-\infty} \frac{5}{x^3-4} = \lim_{x\to-\infty} \frac{\frac{5}{x^3}}{1-\frac{4}{x^3}} = 0$, since the numerator goes to zero and the denominator does not. Here, we divided the numerator and denominator by x^3, the highest power of x found in the fraction.

   (c) $\lim_{x\to\infty} \frac{5x}{x-1} = \lim_{x\to\infty} \frac{5}{1-\frac{1}{x}} = 5$, since the numerator limits to 5 and the denominator limits to 1. Here, we divided the top and bottom of the fraction by x.

   (d) $\lim_{x\to\infty} \frac{7x^2}{4x^2-4x+1} = \lim_{x\to\infty} \frac{7}{4-\frac{4}{x}+\frac{1}{x^2}} = \frac{7}{4}$. We divided the numerator and denominator by x^2.

2. Find the horizontal asymptotes of the following:

   (a) $f(x) = \frac{5x}{x-1}$ has a horizontal asymptote of $y = 5$.

   (b) $f(x) = \frac{7x^2}{4x^2-4x+1}$ has a horizontal asymptote of $y = \frac{7}{4}$.

   (c) $f(x) = \frac{7x^4}{2x+1}$ has no horizontal asymptote, since the function does not limit to a constant (it goes off to infinity).

   (d) $f(x) = \frac{2x+1}{7x^4}$ has a horizontal asymptote of $y = 0$.