Section 9.1, Limits

Let $f(x)$ be a function defined on an open interval containing $x = c$, except perhaps at c. Then,

$$\lim_{x \to c} f(x) = L$$

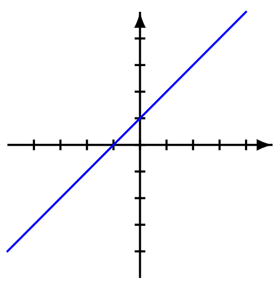
is read "the limit of $f(x)$ as x approaches c equals L." The number L exists if we can make values of $f(x)$ as close to L as we desire by choosing values of x sufficiently close to c. When $f(x)$ does not approach a single finite value L as x approaches c, we say that the limit does not exits (or "DNE").

There is a table with useful properties of limits on page 584 of your textbook.

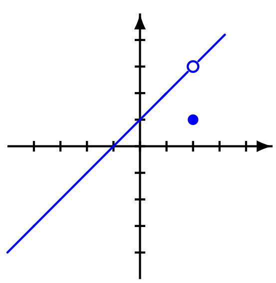
Examples

Find the requested limits for the following graphs. (The blue line in each graph represents $y = f(x)$.)

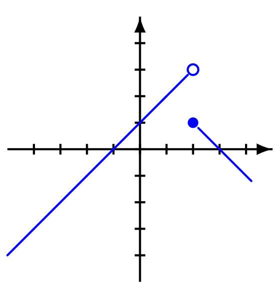
1. $\lim_{x \to 2} f(x) = 3$, since the y-value of the graph approaches 3 as x approaches 2.



2. $\lim_{x \to 2} f(x) = 3$, since the y-value of the graph approaches 3 as x approaches 2 (even though this isn't the value of the function at $x = 2$).



3. $\lim_{x \to 2} f(x)$ DNE, since the function does not approach just one value as x approaches 2.



1 Limits for Polynomials and Rational Functions

If $f(x)$ is a polynomial ($a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ where the a_i's are constant), then $\lim_{x \to c} f(x) = f(c)$ for any real number c.

Examples

1. $\lim_{x\to 2} x^2 - 2x + 3 = 2^2 - 2\cdot 2 + 3 = 3$
2. $\lim_{x\to 3} x + 4 = 3 + 4 = 7$

If two functions, f and g, are both polynomial functions, then $h(x) = \frac{f(x)}{g(x)}$ is called a rational function. To find limits for rational functions, there are three possible scenarios:

1. If $g(c) \neq 0$, then $\lim_{x\to c} h(x) = \frac{f(c)}{g(c)}$.
2. If $g(c) = 0$ and $f(c) \neq 0$, then $\lim_{x\to c} h(x)$ DNE (since we cannot divide by zero). The graph of $h(x)$ will have a vertical asymptote at $x = c$.
3. If $g(c) = 0$ and $f(c) = 0$, then $\lim_{x\to c} h(x)$ has the form $\frac{0}{0}$, which is called the $\frac{0}{0}$ **indeterminate form**. In order to evaluate limits like this, we must first reduce the fraction $\frac{f(x)}{g(x)}$, then try to evaluate it again (The fraction will always reduce in this situation for rational functions, and it might even reduce to a polynomial function, which is simpler to evaluate.).

Examples

1. $\lim_{x\to 3} \frac{x-2}{x+4} = \frac{3-2}{3+4} = \frac{1}{7}$, since the denominator is not zero at $x = 3$.
2. $\lim_{x\to 3} \frac{x^2+2x+9}{x-3}$ DNE, since the denominator is zero at $x = 3$, but the numerator is not zero.
3. $\lim_{x\to 5} \frac{x^2-25}{x-5} = \lim_{x\to 5} \frac{(x-5)(x+5)}{x-5} = \lim_{x\to 5} x + 5 = 5 + 5 = 10$. This was the $\frac{0}{0}$ indeterminate form, so we first reduced the fraction, then evaluated the simpler limit.

2 One-Sided Limits

Sometimes, we only need information on what happens when the x-value approaches a particular value c from either the left or the right side (as opposed to considering what happens as x approaches c from both sides at once, as we have looked at thus far).

The **limit from the right** for a function f is denoted

$$\lim_{x\to c^+} f(x) = L$$

and means that the values of $f(x)$ approach L as x approaches c from the right (the "positive" side).

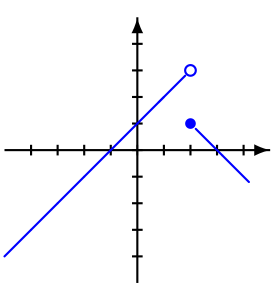
The **limit from the left** for a function f is denoted

$$\lim_{x\to c^-} f(x) = L$$

and means that the values of $f(x)$ approach L as x approaches c from the left (the "negative" side).

Examples

Consider the following graph:



1. $\lim_{x\to 2^+} f(x) = 1$, since $f(x)$ approaches the value 1 when x approaches 2 from the right.
2. $\lim_{x\to 2^-} f(x) = 3$, since $f(x)$ approaches the value 3 when x approaches 2 from the left.

Note that if, for a function f and a real number c, $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$ for some number L, then $\lim_{x\to c} f(x)$ exists and is also equal to L.