Section 14.2, Partial Differentiation

1 Partial Derivatives

If $z = f(x, y)$, then the partial derivative of $f(x, y)$ with respect to x (and many different variations on notation) is

$$z_x = f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

The partial derivative of $f(x, y)$ with respect to y (and notation) is

$$z_y = f_y(x, y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Examples

1. Let $f(x, y) = 2x^2 y + e^x + \ln y$.
   (a) Find $\frac{\partial f}{\partial x}$

   From the definition, note that this is just the "regular" derivative, if we treat y as a constant, so we can use the power rule and the derivative of exponential functions to get $\frac{\partial f}{\partial x} = 4xy + e^x$.

   (b) Find $\frac{\partial f}{\partial y}$

   This time, we will treat x as a constant, so $\frac{\partial f}{\partial y} = 2x^2 + \frac{1}{y}$.
2. We also did #14 from the book as an example.

2 Second Partial Derivatives

As with "normal" derivatives, we can take the second derivative of a function of two variables. As with first partial derivatives, you will be told whether you need to differentiate with respect to x or y (it is possible to combine both). The notation is:

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \text{ (take the partial derivative with respect to } x \text{ twice)}$$
$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \text{ (take the partial derivative with respect to } y \text{ twice)}$$
$$z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \text{ (take the partial derivative with respect to } x\text{, then } y)$$
$$z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \text{ (take the partial derivative with respect to } y\text{, then } x)$$

1. Let $f(x, y) = 2x^2 y + 3xy + \ln(xy) + ye^x$. Find f_{xx}, f_{yy}, f_{xy}, and f_{yx}.

   We will need the first partial derivatives to start, so

$$f_x(x, y) = 4xy + 3y + \frac{y}{xy} + ye^x = 4xy + 3y + x^{-1} + ye^x$$
$$f_y(x, y) = 2x^2 + 3x + y^{-1} + e^x$$

So, the requested second partial derivatives are:

$$\begin{aligned}
f_{xx}(x,y) &= 4y - x^{-2} + ye^x \\
f_{yy}(x,y) &= -y^{-2} \\
f_{xy}(x,y) &= 4x + 3 + e^x \\
f_{yx}(x,y) &= 4x + 3 + e^x
\end{aligned}$$

Note that $f_{xy} = f_{yx}$. This will be the case for most functions, but there are some exceptions, such as when the second partial derivatives are not continuous.

2. For #40, the notation means to evaluate the second partial derivative, z_{xy} at the point $(1, 2)$.