

Section 13.7, Improper Integrals

When we have looked at definite integrals so far, there have been distinct upper and lower bounds for the interval. Here, we will look at what happens when one of the bounds is infinite. If a and b are real numbers, we can find definite integrals with infinite bounds as follows:

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

If the limit is a unique finite number, we say that the integral **converges**. Otherwise, we say that the integral **diverges**.

Note: $\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx$ for any real number a , and converges exactly when both integrals in the sum converge.

Examples

1. $\int_{-\infty}^{-1} \frac{1}{x^2} dx$

$$\int_{-\infty}^{-1} \frac{1}{x^2} dx = \lim_{a \rightarrow -\infty} \int_a^{-1} x^{-2} dx = \lim_{a \rightarrow -\infty} \left(-x^{-1} \Big|_a^{-1} \right) = \lim_{a \rightarrow -\infty} (1 + a^{-1}) = 1$$

2. $\int_1^\infty x e^{-x^2} dx$

$$\begin{aligned} \int_1^\infty x e^{-x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b x e^{-x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{2} e^{-x^2} \Big|_1^b \right) = \lim_{b \rightarrow \infty} \left(-\frac{1}{2} e^{-b^2} + \frac{1}{2} e^{-1} \right) \\ &= \frac{1}{2e} \end{aligned}$$

3. $\int_2^\infty \frac{1}{e^{3x}} dx$

$$\int_2^\infty \frac{1}{e^{3x}} dx = \lim_{b \rightarrow \infty} \int_2^b e^{-3x} dx = \lim_{b \rightarrow \infty} \frac{1}{-3} e^{-3x} \Big|_2^b = \frac{1}{3e^6}$$

4. $\int_1^\infty \frac{1}{\sqrt{x}} dx$

$$\int_1^\infty \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-1/2} dx = \lim_{b \rightarrow \infty} 2x^{1/2} \Big|_1^b = \lim_{b \rightarrow \infty} (2b^{1/2} - 2)$$

Since the first term goes to infinity, the limit is infinite, so this integral diverges.

5. $\int_{-\infty}^\infty e^x dx$

$$\int_{-\infty}^\infty e^x dx = \int_{-\infty}^0 e^x dx + \int_0^\infty e^x dx = \lim_{a \rightarrow -\infty} e^x \Big|_a^0 + \lim_{b \rightarrow \infty} e^x \Big|_0^b = 1 + \text{a term that diverges}$$

Since the second integral in the sum diverges, the original integral diverges as well.

6. Find the area under the curve $f(x) = \frac{1}{x^2\sqrt{x}}$ and to the right of $x = 1$.

$$\int_1^\infty x^{-5/2} dx = -\frac{2}{3} x^{-3/2} \Big|_1^\infty = \frac{2}{3}$$