

Section 13.3, Area Between Two Curves

1 Area Between Curves

If f and g are continuous functions on the interval $[a, b]$ and $f(x) \geq g(x)$ on $[a, b]$, then the area A of the region bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ is:

$$A = \int_a^b (f(x) - g(x)) dx$$

Examples

1. Find the area bounded by $y = x^2$, $y = x$, $x = 0$, and $x = 1$.

On the interval $[0, 1]$, $x \geq x^2$, so

$$A = \int_0^1 (x - x^2) dx = \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = \left(\frac{1}{2} - \frac{1}{3} \right) - 0 = \frac{1}{6}$$

2. Find the area bounded by $f(x) = x^2 + 3$, $g(x) = 2x$, $x = 0$, and $x = 2$.

On the given interval $f(x) \geq g(x)$, so

$$A = \int_0^2 (x^2 + 3 - 2x) dx = \left(\frac{x^3}{3} + 3x - x^2 \right) \Big|_0^2 = \frac{8}{3} + 3 \cdot 2 - 4 - 0 = \frac{14}{3}$$

3. Find the area bounded by $f(x) = 4 - x^2$ and $g(x) = x^2$.

Here, we are not given the interval, so we must find it ourselves. After we graph the functions, we see that there is a bounded (i.e. enclosed) region, so the x -values in that region will give us our interval. To find those x -values, we need to find where the curves intersect:

$$\begin{aligned} 4 - x^2 &= x^2 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

so the needed interval is $[-\sqrt{2}, \sqrt{2}]$.

$$\begin{aligned} A &= \int_{-\sqrt{2}}^{\sqrt{2}} (4 - x^2 - x^2) dx = \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 2x^2) dx \\ &= \left(4x - \frac{2}{3}x^3 \right) \Big|_{-\sqrt{2}}^{\sqrt{2}} = \frac{16}{3}\sqrt{2} \end{aligned}$$

4. Find the area bounded by $f(x) = x^3 - x$ and $g(x) = x$.

Again, we aren't given the interval on which to find the area. When we graph these functions, there are two enclosed regions, so we will need to find the area of each enclosed region, then add them to get the total area. First, we will find where the two curves intersect:

$$\begin{aligned} x^3 - x &= x \\ x^3 - 2x &= 0 \\ x(x^2 - 2) &= 0 \\ x &= 0, \pm\sqrt{2} \end{aligned}$$

$f(x) \geq g(x)$ on $[-\sqrt{2}, 0]$ and $g(x) \geq f(x)$ on $[0, \sqrt{2}]$, so the total area is:

$$\begin{aligned} A &= \int_{-\sqrt{2}}^0 (x^3 - x - x) dx + \int_0^{\sqrt{2}} (x - (x^3 - x)) dx \\ &= \int_{-\sqrt{2}}^0 (x^3 - 2x) dx + \int_0^{\sqrt{2}} (2x - x^3) dx \\ &= \left(\frac{1}{4}x^4 - x^2 \right) \Big|_{-\sqrt{2}}^0 + \left(x^2 - \frac{1}{4}x^4 \right) \Big|_0^{\sqrt{2}} \\ &= 0 - (1 - 2) + (2 - 1) - 0 = 2 \end{aligned}$$

The area in each of the two smaller bounded regions is the same in this example, but this is not always the case. Here, it works because this is an odd function (symmetric about the origin), but this does not hold for general functions.

5. We also set up problem #26 from the homework.

2 Average Value

The **average value** of a continuous function $y = f(x)$ on the interval $[a, b]$ is $\frac{1}{b-a} \int_a^b f(x) dx$.

Examples

1. Find the average value of $f(x) = xe^{x^2}$ from $x = 0$ to $x = 2$.

$$\frac{1}{2-0} \int_0^2 xe^{x^2} dx = \frac{1}{2} \cdot \frac{1}{2} e^{x^2} \Big|_0^2 = \frac{1}{4} (e^4 - 1)$$

2. We did #31 from the book.