Section 13.2, The Definite Integral: The Fundamental Theorem of Calculus

If f is a function on the interval $[a, b]$, then the **definite integral** of f from a to b is:

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+i\frac{b-a}{n}\right)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+(i-1)\frac{b-a}{n}\right)\end{aligned}$$

If f is a continuous function on this interval, then this limit exists, and we say that f is integrable on $[a, b]$.

The Fundamental Theorem of Calculus Let f be a continuous function on a closed interval $[a, b]$. Then the definite integral of f exists on $[a, b]$, and

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any function such that $F'(x) = f(x)$.

Examples

1. $\int_1^2 3x\,dx = \frac{3}{2}x^2\Big|_1^2 = \frac{3}{2}\cdot 2^2 - \frac{3}{2} = \frac{12-3}{2} = \frac{9}{2}$
2. $\int_3^9 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_3^9 = \frac{2}{3}\cdot 9^{3/2} - \frac{2}{3}3^{3/2} = \frac{2}{3}\cdot 27 - \frac{2}{3}\cdot 3\sqrt{3} = 18 - 2\sqrt{3}$
3. $\int_0^4 e^{2x}\,dx = \frac{1}{2}e^{2x}\Big|_0^4 = \frac{1}{2}e^8 - \frac{1}{2}$
4. $\int_1^3 y^{-1}\,dy = \ln|y|\Big|_1^3 = \ln 3 - \ln 1 = \ln 3$

If f is a continuous function on $[a, b]$ and $f(x) \geq 0$, then the area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is $\int_a^b f(x)\,dx$.

Examples

1. We did #37 from the book.
2. (#60 from the book) The rate of production of a new line of products is given by $\frac{dx}{dt} = 200[1 + \frac{400}{(t+40)^2}]$, where x is the number of items produced and t is the number of weeks the products have been in production. How many units were produced in the sixth week?

   The sixth week is from $t = 5$ to $t = 6$, so the total units produced that week is:

$$\begin{aligned}\int_5^6 200[1+400(t+40)^{-2}]\,dt &= 200[t-400(t+40)^{-1}]\Big|_5^6\\ &= 200[6-400(6+40)^{-1}] - 200[5-400(5+40)^{-1}]\\ &\approx 238.647\end{aligned}$$