

## Section 13.1, Area Under a Curve

### 1 Sigma Notation

The Greek letter  $\Sigma$  can be used as shorthand notation for addition. We define the notation as follows:

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n \text{ or}$$
$$\sum_{i=1}^n f(i) = f(1) + f(2) + \cdots + f(n)$$

where the  $a_i$ 's are real numbers, and  $f$  is a function.

**Examples** Calculate each of the following:

1.  $\sum_{i=1}^5 i = 1 + 2 + 3 + 4 + 5 = 15$
2.  $\sum_{i=1}^4 2i = 2 + 4 + 6 + 8 = 20$
3.  $\sum_{i=2}^3 (3i + 1) = (3 \cdot 2 + 1) + (3 \cdot 3 + 1) = 17$
4.  $\sum_{k=1}^6 k^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91$

**Formulas**

1.  $\sum_{i=1}^n 1 = n$
2.  $\sum_{i=1}^n cx_i = c \sum_{i=1}^n x_i$
3.  $\sum_{i=1}^n (x_i + y_i) = \sum_{i=1}^n x_i + \sum_{i=1}^n y_i$
4.  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
5.  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

### 2 Area Under a Curve

In this section, we will be looking at different approaches to approximate the area under a curve on an interval. The basic technique will be to split our interval up into some number of subintervals of equal length (the number of subintervals is typically given in the problem). Then, we will evaluate the given function at either the left or right endpoint of the subinterval (the problem will normally tell you whether to use left or right endpoints), and use these to find the areas of rectangles for each subintervals. Then, we will add the areas of the rectangles. The more rectangles we have, the better our approximation will be.

**Example**

1. Approximate the area under the curve  $y = x^2 + 1$  from  $x = 0$  to  $x = 2$  using 4 equal subintervals. Use the left-hand approximation to approximate the area.

Since we have 4 subintervals, each will have length  $2/4 = \frac{1}{2}$ .



- (a) First rectangle: This will go from  $x = 0$  to  $x = \frac{1}{2}$ . The left endpoint is 0, so the value of the function is  $f(0) = 1$ . The area of this first rectangle is  $\frac{1}{2} \cdot f(0) = \frac{1}{2}$ .
- (b) Second rectangle: This will go from  $x = \frac{1}{2}$  to  $x = 1$ . The left endpoint is  $\frac{1}{2}$ , so the value of the function is  $f(\frac{1}{2}) = \frac{5}{4}$ . The area of this rectangle is  $\frac{1}{2} \cdot f(\frac{1}{2}) = \frac{5}{8}$ .
- (c) Third rectangle: This will go from  $x = 1$  to  $x = \frac{3}{2}$ . The left endpoint is 1, so the value of the function is  $f(1) = 2$ . The area of this rectangle is  $\frac{1}{2} \cdot f(1) = 1$ .
- (d) Fourth rectangle: This will go from  $x = \frac{3}{2}$  to  $x = 2$ . The left endpoint is  $\frac{3}{2}$ , so the value of the function is  $f(\frac{3}{2}) = \frac{13}{4}$ . The area of this rectangle is  $\frac{1}{2} \cdot f(\frac{3}{2}) = \frac{13}{8}$ .

Now, the sum of the areas of the 4 rectangles gives us the approximate area under the curve:  
 Area  $\approx \frac{1}{2} + \frac{5}{8} + 1 + \frac{13}{8} = \frac{15}{4}$ .

2. Approximate the area under the curve  $y = x^2 + 1$  from  $x = 0$  to  $x = 2$ , using 4 subintervals with the right-hand approximation.

This time, we will use Sigma notation to calculate the area.

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^4 (\text{width of rectangle})(\text{height of rectangle}) \\ &= \sum_{i=1}^4 \frac{1}{2} f\left(0 + \frac{1}{2}i\right) = \sum_{i=1}^4 \frac{1}{2} \left(\frac{1}{4}i^2 + 1\right) \\ &= \frac{1}{2} \sum_{i=1}^4 1 + \frac{1}{8} \sum_{i=1}^4 i^2 = \frac{1}{2} \cdot 4 + \frac{1}{8} \cdot \frac{4 \cdot 5 \cdot 9}{6} = \frac{23}{4} \end{aligned}$$

Here, the left-hand approximation gave a smaller answer than the actual area, while the right-hand approximation gave a larger answer. This is not always the case (it depends on the function). As the number of subintervals increases, the approximation becomes more accurate. To find the exact area under the curve  $y = f(x)$  from  $x = a$  to  $x = b$ , we can use:

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{b-a}{n}\right) f\left(a + i\frac{b-a}{n}\right) \text{ using right-hand endpoints} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{b-a}{n}\right) f\left(a + (i-1)\frac{b-a}{n}\right) \text{ using left-hand endpoints} \end{aligned}$$

### Example

Find the area under the curve  $y = x^2 + 1$  from  $x = 0$  to  $x = 2$  using the right-hand approximation.

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2}{n}\right) f\left(i\frac{2}{n}\right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2}{n}\right) \left(\frac{4i^2}{n^2} + 1\right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{8i^2}{n^3} + \frac{2}{n}\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + 2\right) \\ &= \frac{16}{6} + 2 = \frac{14}{3} \end{aligned}$$