Section 11.3, Implicit Differentiation

So far, we have looked at equations involving y's and x's where we can solve for y in terms of x. Then, it is called an **explicit function** of x. However, there are equations involving y's and x's cannot be solved for y. In this case, y is called an **implicit function** of x. Sometimes, we still want to know the rate at which y changes as x changes (that is, $\frac{dy}{dx}$).

For example, the equation of a circle of radius 2 centered at the origin is $x^2 + y^2 = 4$. If we solve this for y, we get $y = \pm\sqrt{4 - x^2}$, which is not a function of x.

To find $\frac{dy}{dx}$ when y is an implicit function of x, we will need a technique called **implicit differentiation**. The steps needed for implicit differentiation for a given equation are:

1. Differentiate both sides of the equation with respect to x. Remember that the chain rule applies, and that the derivative of y with respect to x is $\frac{dy}{dx}$.
2. Solve for $\frac{dy}{dx}$.

Examples

1. Find $\frac{dy}{dx}$ for the following equations:
   (a) $2x^2 + 3y + 7 = 0$

   Taking the derivative of both sides with respect to x, we need the power rule for the first term, and remember that the derivative of y is $\frac{dy}{dx}$. Then, we solve for $\frac{dy}{dx}$.

$$\begin{aligned} 4x + 3\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{4x}{3} \end{aligned}$$

   In this example, we actually could have solved for y, then taking the derivative the "normal" way. This method was just used here for the sake of practice.

   (b) $2x^3 - 7y^2 + x = 9$

$$\begin{aligned} 6x^2 - 14y \cdot \frac{dy}{dx} + 1 &= 0 \\ 14y \cdot \frac{dy}{dx} &= 6x^2 + 1 \\ \frac{dy}{dx} &= \frac{6x^2 + 1}{14y} \end{aligned}$$

   Again, the chain rule was used to get the $\frac{dy}{dx}$ when the derivative of y was needed.

   (c) $4xy + 7x^2 = 0$

   Here, we have a product in the first term, so the product rule will also be needed.

$$\begin{aligned} 4x\frac{dy}{dx} + 4y + 14x &= 0 \\ 4x\frac{dy}{dx} &= -(4y + 14x) \\ \frac{dy}{dx} &= -\frac{4y + 14x}{4x} = -\frac{2y + 7x}{2x} \end{aligned}$$

   We also could have solved this one for y before taking a derivative. If we took that approach, our answer would have looked different at first glance, but in the end it would be the same.

(d) $3x^3y^4 = 2x^2 - 4y$

$$\begin{aligned}
3x^3 \cdot 4y^3 \cdot \frac{dy}{dx} + 3y^4 \cdot 3x^2 &= 4x - 4\frac{dy}{dx} \\
12x^3y^3 \cdot \frac{dy}{dx} + 9x^2y^4 &= 4x - 4\frac{dy}{dx} \\
\frac{dy}{dx}(12x^3y^3 + 4) &= 4x - 9x^2y^4 \\
\frac{dy}{dx} &= \frac{4x - 9x^2y^4}{12x^3y^3 + 4}
\end{aligned}$$

(e) $(x+y)^3 = 2xy$

Here, since we have a quantity raised to a power, we will need the general power rule:

$$\begin{aligned}
3(x+y)^2\left(1 + \frac{dy}{dx}\right) &= 2x\frac{dy}{dx} + 2y \\
\frac{dy}{dx}(3(x+y)^2 - 2x) &= 2y - 3(x+y)^2 \\
\frac{dy}{dx} &= \frac{2y - 3(x+y)^2}{3(x+y)^2 - 2x}
\end{aligned}$$

2. Find $\frac{dy}{dx}$ at the point $(\frac{27}{4}, -3)$ if $2y^2 - 4x + 9 = 0$.

First, we will carry out implicit differentiation, then put in the given x-value and y-value:

$$\begin{aligned}
4y\frac{dy}{dx} - 4 &= 0 \\
\frac{dy}{dx} &= \frac{1}{y} \\
\text{At the given point, } \frac{dy}{dx} &= -\frac{1}{3}
\end{aligned}$$

3. Let p and q satisfy $p^2q = 4p - 2$.

(a) Find $\frac{dp}{dq}$.

Here, we will treat p as y and q as x, so:

$$\begin{aligned}
p^2 + q \cdot 2p\frac{dp}{dq} &= 4\frac{dp}{dq} \\
\frac{dp}{dq}(2pq - 4) &= -p^2 \\
\frac{dp}{dq} &= -\frac{p^2}{2pq - 4}
\end{aligned}$$

(b) Find $\frac{dq}{dp}$.

Note that $\frac{dp}{dq}$ and $\frac{dq}{dp}$ are reciprocals, so we only need to find:

$$\frac{dq}{dp} = \frac{1}{\text{The answer to the previous part}} = -\frac{2pq - 4}{p^2}$$

4. (#56 from the book) Suppose that the number of mosquitoes N (in thousands) in a certain swampy area near a community is related to the number of pounds of insecticide x sprayed on the nesting area according to $Nx - 10x + N = 300$. Find the rate of change of N with respect to x when 49 pounds of insecticide is used.

In other words, we want to find $\frac{dN}{dx}$ when $x = 49$. First, we will use implicit differentiation to find the needed rate of change:

$$
\begin{aligned}
N + x\frac{dN}{dx} - 10 + \frac{dN}{dx} &= 0 \\
\frac{dN}{dx}(x+1) &= 10 - N \\
\frac{dN}{dx} &= \frac{10-N}{x+1}
\end{aligned}
$$

To evaluate this, we need both x and N (we only have x), so we will solve the original equation for N when $x = 49$:

$$
\begin{aligned}
49N - 10 \cdot 49 + N &= 300 \\
50N &= 300 + 490 = 790 \\
N &= \frac{790}{50} = \frac{79}{5}
\end{aligned}
$$

Therefore,

$$
\frac{dN}{dx} = \frac{10-N}{x+1} = \frac{10 - \frac{79}{5}}{49+1} = -\frac{29}{250}
$$