Section 11.1, Derivatives of Logarithmic Functions

1 Quick Review of Logarithms

For a constant a with $a > 0$ and $a \neq 1$, recall that for $x > 0$, $y = \log_a x$ if $a^y = x$. For example, $\log_2 8 = 3$ since $2^3 = 8$ and $\log_3 \frac{1}{3} = -1$ since $3^{-1} = \frac{1}{3}$.

$\log_e x$ is frequently denoted $\ln x$, and called the "natural logarithm." ($e \approx 2.71828$). $\log_{10} x$ is frequently denoted $\log x$, and is called the "common logarithm."

Properties of Logarithms:

1. $\log_a a^x = x$
2. $a^{\log_a x} = x$
3. $\log_a(xy) = \log_a x + \log_a y$
4. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
5. $\log_a x^n = n \log_a x$
6. $\log_a x = \frac{\log_b x}{\log_b a}$. This is often called the "change-of-base formula."

Examples

1. Evaluate each of the following:
   (a) $\log_8 64 = 2$
   (b) $\log_8 4 = \frac{2}{3}$
   (c) $\log_9 3 = \frac{1}{2}$
   (d) $\log_{42} 1 = 0$
   (e) $\log_4 0$ is undefined since we cannot take the logarithm of zero or negative numbers.
   (f) $\log_1 7$ is undefined since the base cannot be one (or zero or negative numbers).
   (g) $\log_2 \frac{1}{4} = -2$
   (h) $\log_{1/4} 64 = -3$
2. Write each of the following using only one logarithm:
   (a) $\log_4 x + \log_4(x+2) - \log_4(x-2) = \log_4 \frac{x(x+2)}{x-2}$
   (b) $2\log_7 x - 3\log_7(x-1) + 2 = \log_7 x^2 - \log_7(x-1)^3 + \log_7 49 = \log_7 \frac{49x^2}{(x-1)^3}$
3. Write each of the following in terms of $\log_5 a$, $\log_5 b$, and $\log_5 c$.
   (a) $\log_5 \frac{ab}{c} = \log_5 a + \log_5 b - \log_5 c$
   (b) $\log_5 \sqrt{ac} = \frac{1}{2}\log_5 a + \frac{1}{2}\log_5 c$
   (c) $\log_5 \frac{a}{b^2} = \log_5 a - 2\log_5 b$
4. Solve $2^x = 9$.

   We can start by taking the logarithm of both sides. We can choose to use any base, as long as it's the same on both sides. Normally, you should choose either the based used in the exponential part due to cancellation that will occur, or base e or 10 since they are frequently available on calculators. Here, I'll use base e.

$$\begin{aligned} \ln 2^x &= \ln 9 \\ x \ln 2 &= \ln 9 \\ x &= \frac{\ln 9}{\ln 2} \approx 3.1699 \end{aligned}$$

5. \$2000 is invested into a bank account earning an APR of 3%, compounded monthly. How many years will it take for the investment to reach \$3000?

   From compound interest equations, we know that for Y years,

$$\begin{aligned} 3000 &= 2000\left(1+\frac{.03}{12}\right)^{12Y} \\ 3000 &= 2000 \cdot 1.0025^{12Y} \\ 1.5 &= 1.0025^{12Y} \\ \ln 1.5 &= \ln 1.0025^{12Y} \\ \ln 1.5 &= 12Y \ln 1.0025 \\ Y &= \frac{\ln 1.5}{12 \ln 1.0025} \approx 13.53239 \end{aligned}$$

2 Derivatives

If $y = \ln x$, then $y' = \frac{1}{x}$. If $y = \ln g(x)$, then $y' = \frac{g'(x)}{g(x)}$ by the chain rule.

Examples

Find the derivative of each of the following:

1. $f(x) = 5 \ln x$

$$f'(x) = 5 \cdot \frac{1}{x} = \frac{5}{x}$$

2. $f(x) = \ln(14x^5)$

$$f'(x) = \frac{14 \cdot 5x^4}{14x^5} = \frac{5}{x}$$

   Note: This is the same derivative as the last example, since using properties of derivatives, you can show that the two functions differ by a constant.

3. $g(x) = 4 + \ln(2x^2 - 4)^3$

$$g'(x) = 0 + \frac{3(2x^2-4)^2 \cdot 4x}{(2x^2-4)^3} = \frac{12x}{2x^2-4} = \frac{6x}{x^2-2}$$

4. $h(t) = t^2 + \log_2 t^3$

   Since this is not a natural logarithm, we cannot go directly from the formula. However, we can use the change-of-base formula first, so we rewrite $h(t) = t^2 + \frac{\ln t^3}{\ln 2}$, then we can use the natural logarithm derivative formula:

$$h'(t) = 2t + \frac{3t^2}{t^3 \ln 2} = 2t + \frac{3}{t \ln 2}$$

5. $y = (\ln x)^{-2}$

   Since the ln is being raised to a power, we need the general power rule to find y':

$$y' = -2 \cdot (\ln x)^{-3} \cdot \frac{1}{x} = -\frac{2}{x(\ln x)^3}$$

6. We also did #49 from the homework in class.