Section 10.4, Applications of Maxima and Minima

Examples

1. A farmer wants to enclose a rectangular plot of land that must contain 3600 square feet. What should the dimensions be if he needs to minimize the fencing needed to enclose the area?

   First, we need to define any needed variables and then set up an equation (or, possibly a few equations). Since the plot of land is rectangular, let x represent its width and y represent its length. We want to minimize the perimeter, or

   $$P = 2x + 2y$$

   Unfortunately, the perimeter is a function of 2 variables, so we need to find a way to eliminate one of the variables. This is where the given area comes in:

   $$\text{Area} = xy = 3600 \text{ so } y = \frac{3600}{x}$$

   Substituting this into our perimeter equation:

   $$P = 2x + 2 \cdot \frac{3600}{x} = 2x + 7200x^{-1}$$

   To minimize this, we need the critical values, so:

   $$\begin{aligned} P' &= 2 - 7200x^{-2} \\ 0 &= 2 - 7200x^{-2} \\ x^2 &= 3600 \\ x &= 60 \text{ (we can ignore the } x = -60 \text{ solution, since the width must be positive)} \end{aligned}$$

   Now, using $y = \frac{3600}{x}$ to find the length of the rectangle, we know that the minimum fencing happens when the plot of land is $60' \times 60'$.

2. A vacationer on an island 8 miles offshore from a point that is 48 miles from town must travel to town occasionally. The vacationer has a boat capable of traveling 30 mph and can go by auto along the coast at 55 mph. At what point should the car be left to minimize the time it takes to get to town? (See the diagram for #32 on page 741 of the textbook.)

   The book assigns x so that $48 - x$ is the distance the car should be left from the town. Using the equation distance = rate × time, and solving for the time, we know that the time driven will be:

   $$\text{time driven} = \frac{\text{distance driven}}{\text{car's speed}} = \frac{48 - x}{55}$$

   The time spent on the boat will be calculated in a similar fashion, but we need to find the distance the boat travels using the Pythagorean Theorem ($a^2 + b^2 = c^2$), so

   $$8^2 + x^2 = c^2 \text{ so } c = \sqrt{64 + x^2} = (64 + x^2)^{1/2}$$

   So,

   $$\text{time by boat} = \frac{\text{distance on boat}}{\text{boat's speed}} = \frac{(64 + x^2)^{1/2}}{30}$$

   Therefore, the total time T spent in traveling to time is:

   $$T = \frac{48}{55} - \frac{x}{55} + \frac{(64 + x^2)^{1/2}}{30}$$

This is the function that we are asked to minimize. Finding the critical values, we see that:

$$
\begin{aligned}
T' &= -\frac{1}{55} + \frac{1}{2} \cdot \frac{(64+x^2)^{-1/2}}{30} \cdot 2x = -\frac{1}{55} + \frac{x}{30(64+x^2)^{1/2}} \\
0 &= -\frac{1}{55} + \frac{x}{30(64+x^2)^{1/2}} \\
55x &= 30(64+x^2)^{1/2} \\
[11x]^2 &= [6(64+x^2)^{1/2}]^2 \\
121x^2 &= 36(64+x^2) \\
121x^2 &= 2304 + 36x^2 \\
x^2 &= \frac{2304}{85} \\
x &= \sqrt{\frac{2304}{85}} \approx 5.206
\end{aligned}
$$

Therefore, the car should be left 5.206 miles from the closest point on the shore to the island (or, 42.794 miles from town).

3. We also set up problem #17.