Section 10.3, Optimizing in Business and Economics

1 Absolute Extrema

The value $f(c)$ is the **absolute maximum** of f if $f(c) \geq f(x)$ for all x in the interval of interest.

The value $f(c)$ is the **absolute minimum** of f if $f(c) \leq f(x)$ for all x in the interval of interest.

If a function f is continuous on a closed interval $[a, b]$, it will have an absolute maximum and an absolute minimum on that interval. These will occur either at a critical value of f, or at an endpoint of the interval.

Example

Find the absolute maximum and minimum of $f(x) = x^3 - 3x + 5$ on the interval $[-5, 2]$.

First, we need the critical values:

$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ 0 &= 3x^2 - 3 = 3(x-1)(x+1) \\ x &= \pm 1 \end{aligned}$$

Now, we need only compare the value of the function f at the critical values, as well as the endpoints of $[-5, 2]$.

$$\begin{aligned} f(-5) &= -105 \\ f(-1) &= 7 \\ f(1) &= 3 \\ f(2) &= 7 \end{aligned}$$

Then, -105 is the absolute minimum and occurs at $x = -5$. The absolute maximum is 7 and occurs at both $x = -1$ and $x = 2$.

2 Application: Average Cost

The **average cost per unit** is:

$$\overline{C} = \frac{C(x)}{x},$$

where $C(x)$ is the total cost when x units are produced.

Example

The total cost function for a product is $C(x) = 300 + 10x + 0.03x^2$. Find the number of units that will result in a minimum average cost per unit. What is this minimum average cost?

First, we need to calculate the average cost function:

$$\overline{C} = \frac{300 + 10x + 0.03x^2}{x} = 300x^{-1} + 10 + 0.03x$$

Now, we need to minimize this function, so we must find the critical values (The fraction for $\overline{C}$ was reduced to make taking the derivative easier. This way, we can avoid the quotient rule, and just use the power rule.).

$$\begin{aligned}
\overline{C}' &= -300x^{-2} + 0.03 \\
0 &= -300x^{-2} + 0.03 \\
x^2 &= \frac{300}{0.03} = 10,000 \\
x &= \pm 100
\end{aligned}$$

Since x refers to the number of units produced, it cannot be negative. Therefore, the only critical value we need to check is $x = 100$. Using either the first or second derivative test, we see that $x = 100$ indeed gives a minimum of $\overline{C}$. Then, the minimum average cost is:

$$\overline{C} = 300 \cdot 100^{-1} + 10 + 0.03 \cdot 100 = 16$$