

## Section 10.2, Concavity: Points of Inflection

### 1 Concavity and Points of Inflection

The graph of a curve  $f$  is said to be **concave up** on an interval if  $f'(x)$  is increasing on that interval (i.e.  $f''(x) > 0$ ).

A curve  $f$  is said to be **concave down** on an interval if  $f'(x)$  is decreasing on that interval (i.e.  $f''(x) < 0$ ).

A point where the concavity of a function changes is called a **point of inflection**. At these points, either  $f''(x) = 0$  or  $f''(x)$  is undefined.

#### Example

Find the point(s) of inflection of  $f(x) = x^3 - 2x^2 + 4$ .

We first need to find where the second derivative is zero (for polynomials, the second derivative is always defined), then verify that the concavity changes at that point.

$$\begin{aligned}f'(x) &= 3x^2 - 4x \\f''(x) &= 6x - 4 \\0 &= 6x - 4 \\x &= \frac{2}{3}\end{aligned}$$

The second derivative is negative to the left of  $x = \frac{2}{3}$  and positive to the right, so there is a point of inflection here (since it is a point, we need the value of the function, too, so we calculate  $f(\frac{2}{3})$ ). The point of inflection is  $(\frac{2}{3}, \frac{92}{27})$ .

### 2 The Second Derivative Test

The second derivative test is used to determine whether there is a maximum or minimum at a given critical value. It is more restrictive than the first derivative test, but is quicker to use for many functions.

For the Second Derivative Test, we first need to find the critical values of  $f$  using the techniques from the previous section. Let  $x = x_0$  represent a critical value where  $f'(x_0) = 0$ . Then, we calculate  $f''(x_0)$ , and it will fall into one of the following categories:

1. If  $f''(x_0) > 0$ , then  $f(x_0)$  is a relative minimum.
2. If  $f''(x_0) < 0$ , then  $f(x_0)$  is a relative maximum.
3. If  $f''(x_0) = 0$  or  $f''(x_0)$  is undefined, then the second derivative does not provide us with enough information, so we must use the first derivative test.

#### Examples

Find the relative maxima and minima of the following functions, as well as point(s) of inflection, then use this information to sketch the graph.

1.  $f(x) = x^3 - 2x^2 + x - 2$

First, we find the critical values:

$$\begin{aligned}f'(x) &= 3x^2 - 4x + 1 \\0 &= 3x^2 - 4x + 1 = (3x - 1)(x - 1) \\x &= \frac{1}{3}, 1\end{aligned}$$

To use the second derivative test, we need the second derivative:

$$f''(x) = 6x - 4$$

Now, we evaluate the second derivative at each of our critical values:

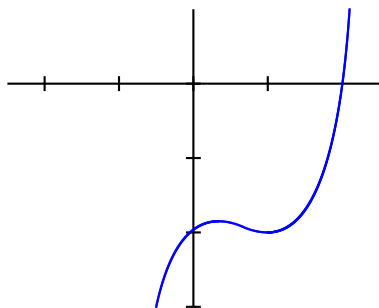
$$\begin{aligned}f''\left(\frac{1}{3}\right) &= 2 - 4 = -2 < 0 \\f''(1) &= 6 - 4 = 2 > 0\end{aligned}$$

This tells us that there is a relative maximum at  $x = \frac{1}{3}$  and a relative minimum at  $x = 1$ . The actual relative maximum point is  $(\frac{1}{3}, -\frac{50}{27})$  and the relative minimum point is  $(1, -2)$ .

To find the point(s) of inflection, we set the second derivative equal to zero:

$$\begin{aligned}0 &= 6x - 4 \\x &= \frac{2}{3}\end{aligned}$$

Since the second derivative changes from negative to positive at  $x = \frac{2}{3}$ , the point of inflection is  $(\frac{2}{3}, -\frac{52}{27})$ .



In cases like this, when the critical points are close together, be sure to show detail in the “interesting” part of the graph. This can be done by zooming in on the part of the graph that contains critical points (as above), labeling the graph, etc.

2. We also started #19 from the homework assignment.