

Section 10.1, Relative Maxima and Minima: Curve Sketching

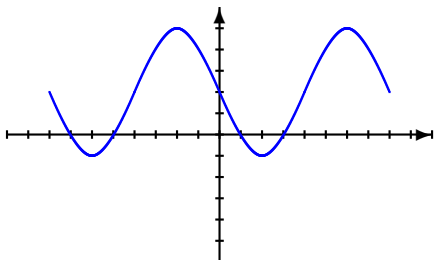
1 Increasing and Decreasing Functions

We say that a function $f(x)$ is **increasing** on an interval if the values of f increase as x increases (i.e. $f'(x) > 0$).

A function $f(x)$ is **decreasing** on an interval if the values of f decrease as x increases (i.e. $f'(x) < 0$).

Example

The blue curve represents the graph of $y = f(x)$, for some function f .



The function is increasing on the intervals $(-6, -2)$ and $(2, 6)$, and is decreasing on the intervals $(-2, 2)$, $(-8, -6)$, and $(6, 8)$. From the graph, we do not know what happens for $x < -8$ or $x > 8$.

2 Relative Maxima and Minima

The point $(x_1, f(x_1))$ is a **relative maximum** point of the function f if there is an interval around x_1 on which $f(x_1) \geq f(x)$ for all x in that interval. We say that the relative maximum occurs at $x = x_1$ and the relative maximum is $f(x_1)$.

Likewise, the point $(x_2, f(x_2))$ is a **relative minimum** point of the function f if there is an interval around x_2 on which $f(x_2) \leq f(x)$ for all x in that interval. We say that the relative minimum occurs at $x = x_2$ and the relative minimum is $f(x_2)$.

On the graph above, the points where the relative maxima and minima occur have horizontal tangent lines, so $f'(x) = 0$ at these points.

In general, if f has a relative maximum or a relative minimum at $x = x_0$, either $f'(x_0) = 0$ or $f'(x_0)$ is undefined. Such values of x are called **critical values**.

Example

Find the critical value(s) of the function $f(x) = x^3 - 2x^2 + 1$.

We are asked to find the values of x for which $f'(x) = 0$ or $f'(x)$ is undefined.

$$\begin{aligned}f'(x) &= 3x^2 - 4x \\0 &= 3x^2 - 4x = x(3x - 4) \\x &= 0, \frac{4}{3}\end{aligned}$$

3 Finding Relative Maxima and Minima: The First Derivative Test

To carry out the first derivative test for a function f , we need to find the critical values first. Let x_0 be a critical value of f .

Next, find an interval (a, b) in which x_0 is the only critical value. We need f to be continuous on the interval (a, b) . Then, our function will fall into one of the following categories:

1. On (a, b) , if $f'(x)$ is negative to the left of $x = x_0$ and positive to the right of $x = x_0$, then $f(x_0)$ is a relative minimum. This happens because the function is decreasing as it approaches $x = x_0$ and increases afterward.
2. On (a, b) , if $f'(x)$ is positive to the left of $x = x_0$ and negative to the right of $x = x_0$, then $f(x_0)$ is a relative maximum. This happens because the function is increasing as it approaches $x = x_0$ and then decreases.
3. If $f'(x)$ has the same sign to the left and right of $x = x_0$ within the interval (a, b) , then $f(x_0)$ is neither a relative maximum nor minimum of f . If, in addition $f'(x_0) = 0$, then $(x_0, f(x_0))$ is called a **horizontal point of inflection** of f .

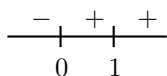
Examples

1. Find the relative maxima and minima, and horizontal points of inflection of $f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + 1$. Also state where the function is increasing and decreasing. Then use this information to sketch the graph of the function.

First, we need to find the critical values of f , so we calculate the first derivative, set it equal to zero, and solve for x :

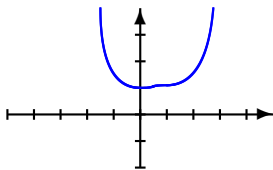
$$\begin{aligned} f'(x) &= x^3 - 2x^2 + x \\ 0 &= x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x - 1)^2 \\ x &= 0, 1 \end{aligned}$$

Testing the sign of the first derivative, we get that:



There is a relative minimum at $x = 0$ and a horizontal point of inflection at $x = 1$. We were asked for points, not just the x -values, so we calculate $f(0)$ and $f(1)$ to get that the relative minimum point is $(0, 1)$ and the horizontal point of inflection is $(1, \frac{13}{12})$.

Reusing the information from the above number line, we get that the function is decreasing on the interval $(-\infty, 0)$ and increasing on the intervals $(0, 1)$ and $(1, \infty)$.



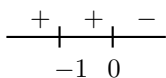
In words, the graph decreases until it reaches the point $(0, 1)$. It then increases until it levels off at $(1, \frac{13}{12})$, then continues to increase.

2. Find the critical values where the relative maxima and minima of $y = -\frac{1}{8}x^8 - \frac{2}{7}x^7 - \frac{1}{6}x^6$ occur.

First, we find the derivative and set it equal to zero to find the critical values:

$$\begin{aligned} y' &= -x^7 - 2x^6 - x^5 \\ 0 &= -x^5(x^2 + 2x + 1) = -x^5(x + 1)^2 \\ x &= -1, 0 \end{aligned}$$

We then find the sign of the derivative around the critical values:



Therefore, there is a relative maximum at $x = 0$.

Some notes on the First Derivative Test and Relative Maxima and Minima:

1. Since this test requires that f is continuous on an interval, this does not work for certain critical values. For example, if the function has a vertical asymptote at $x = x_0$, then $f'(x_0)$ will also be undefined, so $x = x_0$ is a critical value. However, the function does not meet the criteria for this test in an interval around $x = x_0$ since it isn't continuous.
2. When finding whether the first derivative is positive or negative in each interval, remember that you do not need the exact value of the derivative. When finding the critical values, you probably factored the derivative, so you can use that factoring to find whether each factor is positive or negative on that interval, then reason whether the product is positive or negative with basic properties of multiplication. This helps to simplify the calculation a bit and can save time.
3. Read the questions that you are answering carefully. Some ask for the location where a minimum or maximum occurs, so you would just give an x -value. Other questions ask for the actual minimum or maximum of the function, so you would give the actual value of the function at the appropriate critical value. Some problems ask for both.
4. "Minima" is the plural of "minimum." "Maxima" is the plural of "maximum." Please use these words appropriately.