Handout: Exponential and Logarithmic Functions
Math 1100-4
Thursday, January 19, 2012

Properties of Exponents

1. $a^0 = 1$ if $a \neq 0$ (0^0 is undefined)
2. $a^1 = a$
3. $a^m \cdot a^n = a^{m+n}$
4. $\dfrac{a^m}{a^n} = a^{m-n}$
5. $(ab)^m = a^m b^m$
6. $(a^m)^n = a^{mn}$
7. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$
8. $a^{-m} = \dfrac{1}{a^m}$
9. $\left(\dfrac{a}{b}\right)^{-m} = \left(\dfrac{b}{a}\right)^m$
10. $a^{1/n} = \sqrt[n]{a}$

Examples

1. Write the following expressions using exponents, leaving neither fractions nor radicals in your final answer:
   (a) $\sqrt{x} = x^{1/2}$
   (b) $\dfrac{1}{x^3} = x^{-3}$
   (c) $\dfrac{3}{\sqrt[6]{x^5}} = 3x^{-5/6}$
   (d) $(\sqrt[3]{x})^6 = (x^{1/3})^6 = x^{(1/3)\cdot 6} = x^2$
2. Simplify each of the following expressions, and write your answer using only positive exponents:
   (a) $\dfrac{x^3 y^{-4}}{x^7 y^0} = \dfrac{1}{x^4 y^4}$
   (b) $z^3 \cdot \left(\dfrac{x^2}{z^8}\right)^4 = z^3 \cdot \dfrac{x^8}{z^{32}} = \dfrac{x^8}{z^{29}}$

More details and examples of exponential functions can be found in Section 5.1 of your textbook.

Logarithms
If a is a constant satisfying $a > 0$ and $a \neq 1$, then the logarithm with base a of x is denoted $\log_a x$, and is defined by $\log_a x = y$ if $a^y = x$. In other words, the logarithm with base a of x answers the question, "a raised to what power is x?"

Frequently we will use base $e \approx 2.71828$, so we have special notation and terminology for this: $\ln x = \log_e x$. This is called the natural logarithm of x.

Examples

1. $\log_3 9 = 2$ since $3^2 = 9$.
2. $\log_{16} 4 = \frac{1}{2}$ since $16^{1/2} = 4$.
3. $\log_2 \frac{1}{2} = -1$ since $2^{-1} = \frac{1}{2}$.
4. $\ln e^2 = 2$ since $e^2 = e^2$.

Properties of Logarithms
If $a > 0$, $a \neq 1$,

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a (xy) = \log_a x + \log_a y$
4. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
5. $\log_a x^n = n \log_a x$
6. $\log_a \dfrac{1}{y} = -\log_a y$
7. $\log_a x = \dfrac{\log_b x}{\log_b a}$. This is often called the "change-of-base formula."
8. $\log_a a^n = n$
9. $a^{\log_a x} = x$. These last two properties say that logarithmic and exponential functions with the same base are inverse functions. In other words, they cancel each other out if the bases are the same.

Examples

1. $\log_{42} 1 = 0$
2. $\log_7 7^x = x$
3. $\log_4 8 = \dfrac{\log_2 8}{\log_2 4} = \dfrac{3}{2}$

More details and examples of logarithmic functions can be found in Section 5.2 of your textbook.