Section 6.4, Vectors and Dot Products

Homework: 6.4 #1–33 odds, 47, 49, 51, 71, 73

Those of you who need to take a Linear Algebra course will see this topic in much more detail in a later semester.

1 The Dot Product

The **dot product** of two vectors, $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Note that the result is a real number, not a vector.

Examples

1. $\langle 2, 3 \rangle \cdot \langle 1, 4 \rangle = 2 \cdot 1 + 3 \cdot 4 = 14$
2. $\langle -1, 2 \rangle \cdot \langle 5, 8 \rangle = -1 \cdot 5 + 2 \cdot 8 = 11$

Properties of Dot Products

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. In other words, the dot product is commutative.
2. $\mathbf{0} \cdot \mathbf{u} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$. In other words, the dot product is distributive.
4. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$
5. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$ if k is a real number.

Examples
Let $\mathbf{u} = \langle 3, -2 \rangle$, $\mathbf{v} = \langle 4, 2 \rangle$, and $\mathbf{w} = \langle 1, -5 \rangle$. Find each of the following:

1. $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{9 + 4} = \sqrt{13}$
2. $(2\mathbf{u}) \cdot \mathbf{v} = \langle 6, -4 \rangle \cdot \langle 4, 2 \rangle = 24 - 8 = 16$
3. $\mathbf{v} \cdot (\mathbf{u} - \mathbf{w}) = \langle 4, 2 \rangle \cdot \langle 2, 3 \rangle = 8 + 6 = 14$

2 Angles Between Vectors

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

For these types of problems, we normally want $0 \le \theta \le 180^\circ$, so we will find an angle in either Quadrant I or II.

Example
Find the angle between $\mathbf{u} = \langle 3, 3\sqrt{3} \rangle$ and $\mathbf{u} = \langle -3, 3\sqrt{3} \rangle$.

For the formula, we need to find the dot product of the two vectors, as well as their magnitudes:

$$\mathbf{u} \cdot \mathbf{v} = \langle 3, 3\sqrt{3} \rangle \cdot \langle -3, 3\sqrt{3} \rangle = -9 + 27 = 18$$

$$\|\mathbf{u}\| = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{36} = 6$$

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + (3\sqrt{3})^2} = \sqrt{36} = 6,$$

so

$$\cos\theta = \frac{18}{6\cdot 6} = \frac{1}{2}$$
$$\theta = 60^\circ = \frac{\pi}{3}$$

If $\mathbf{u}\cdot\mathbf{v} = 0$, we say that the vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** or **perpendicular**. The angle made between the vectors is 90°.

If $\cos\theta = \pm 1$, we say that the vectors are **parallel**. This happens when the one vector is a multiple of the other when they are written in component form.

Examples
Are the following vectors parallel, orthogonal, or neither?

1. $\langle 4, 1\rangle$, $\langle -2, 8\rangle$

   We can start by finding the dot product of the two vectors. If it is zero, we know the vectors are orthogonal. If the dot product isn't zero, we can go on to calculate the cosine of the angle between the vectors (Since the dot product is needed for the formula, this is still a very direct approach.).

   $$\langle 4, 1\rangle \cdot \langle -2, 8\rangle = 4\cdot(-2) + 1\cdot 8 = 0,$$

   so the vectors are orthogonal.

2. $\langle 2, -5\rangle$, $\langle -4, 10\rangle$

   The dot product of the two vectors is

   $$\langle 2, -5\rangle \cdot \langle -4, 10\rangle = -8 - 50 = -58,$$

   so we know that the vectors are not orthogonal. To determine whether they are parallel or not, we can calculate the cosine of the angle between them. For this, we need the magnitudes of the vectors:

   $$\|\langle 2, -5\rangle\| = \sqrt{4+25} = \sqrt{29}$$
   $$\|\langle -4, 10\rangle\| = \sqrt{16+100} = \sqrt{116} = 2\sqrt{29}$$

   Then, the angle θ between the two vectors satisfies:

   $$\cos\theta = \frac{-58}{\sqrt{29}\cdot 2\sqrt{29}} = -1,$$

   so the vectors are parallel.

3. $2\hat{\imath} + 3\hat{\jmath}$, $-\hat{\imath} - 2\hat{\jmath}$

   The dot product of the vectors is:

   $$(2\hat{\imath} + 3\hat{\jmath}) \cdot (-\hat{\imath} - 2\hat{\jmath}) = -2 - 6 = -8,$$

   so the vectors are not orthogonal. The magnitudes of the vectors is:

   $$\|2\hat{\imath} + 3\hat{\jmath}\| = \sqrt{4+9} = \sqrt{13}$$
   $$\| -\hat{\imath} - 2\hat{\jmath}\| = \sqrt{1+4} = \sqrt{5}$$

   The cosine of the angle θ between the vectors is:

   $$\cos\theta = \frac{-8}{\sqrt{13}\cdot\sqrt{5}} = -\frac{8}{\sqrt{65}},$$

   which isn't equal to either positive or negative one, so these two vectors are not parallel either.

   Note: Since $-\frac{8}{\sqrt{65}}$ is fairly close to -1, you may not be able to tell whether these two vectors are parallel or not just by graphing them.

3 Work

In general, Work = Force $\times$ Distance. However, what happens when not all of the force that is applied is in the same direction as the direction of motion? For example, how can we find what work is done by the force of gravity when an object is rolling down a ramp, instead of falling straight down?

The **work** W done by a force vector $\mathbf{F}$ acting along the vector $\overrightarrow{PQ}$ is given by

$$W = \mathbf{F} \cdot \overrightarrow{PQ}$$

Example
A force of 50 pounds exerted at an angle of 45° is required to pull a full wagon along a level sidewalk. The wagon is pulled 30 feet. Determine the work that is done.

The vector for the distance is $\langle 30, 0\rangle$. The vector for the force is $\langle 50\cos 45°, 50\sin 45°\rangle$, so the work done is

$$W = \langle 50\cos 45°, 50\sin 45°\rangle \cdot \langle 30, 0\rangle = 150\cos 45° = 75\sqrt{2} \text{ ft-lbs}$$