Section 6.3, Vectors in the Plane

Homework: 6.3 #3–9 odds, 21–37 odds, 53, 55

1 Introduction to Vectors

A **vector**, **v**, is a directed line segment. If it starts at a point P and ends at a point Q, it is written as $\mathbf{v} = \overrightarrow{PQ}$. Due to limitations of writing on the board in class, I will normally use $\vec{v}$ instead of the boldface **v**.

Two vectors, **u** and **v** are equal if they have the same magnitude (length) and direction. For example, if we have the points $P = (0,0)$, $Q = (4,-2)$, $R = (1,3)$, and $S = (5,1)$, then $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{RS}$ are equal, since they both go 4 units to the right and 2 units down. One benefit of this is that, for any vector, we can move the initial point to the origin and use only one set of coordinates to describe the vector.

If a vector **v** has an initial point at the origin and its terminal point is (v_1, v_2), it is in **standard position** and its **component form** is $\mathbf{v} = \langle v_1, v_2 \rangle$. The vector $\langle 0, 0 \rangle$ is called the **zero vector**.

The component form of the vector **v** with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is

$$\mathbf{v} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle$$

Its **magnitude** (length) is

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}$$

(This formula can be thought of as a restatement of either the distance formula or the Pythagorean theorem.)

If $\|\mathbf{v}\| = 1$, **v** is called a **unit vector**. $\hat{\imath} = \langle 1, 0 \rangle$ represents a unit vector in the positive x-direction. $\hat{\jmath} = \langle 0, 1 \rangle$ represents a unit vector in the positive y-direction.

$\|\mathbf{v}\| = 0$ exactly when **v** is the zero vector.

Examples
Graph the following vectors and find the component form and magnitude of the vector.

1. Initial point: $(3, 7)$, Terminal point: $(-1, 10)$

   The component form of the vector is

   $$\mathbf{v} = \langle -1 - 3, 10 - 7 \rangle = \langle -4, 3 \rangle$$

   The magnitude is

   $$\|\mathbf{v}\| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$$

   (The vector was graphed in class.)

2. Initial point: $(-2, -3)$, Terminal point: $(5, 1)$

   The component form of the vector is

   $$\mathbf{v} = \langle 5 - (-2), 1 - (-3) \rangle = \langle 7, 4 \rangle$$

   The magnitude is

   $$\|\mathbf{v}\| = \sqrt{7^2 + 4^2} = \sqrt{65}$$

   (The vector was graphed in class.)

2   Mathematical Operations with Vectors

Let $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and k be a real number. Then,

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$$
$$k\mathbf{u} = \langle k \cdot u_1, k \cdot u_2 \rangle$$
$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle = \mathbf{u} + (-1)\mathbf{v}$$

There is a table with some properties of vector addition and scalar multiplication on page 451.

If k is positive, note that $k\mathbf{u}$ points in the same direction as u.

Examples

1. Calculate and simplify each of the following expressions:

    (a) $\langle 3, 2 \rangle + \langle 7, 1 \rangle = \langle 10, 3 \rangle$

    (b) $4\langle -1, 3 \rangle = \langle -4, 12 \rangle$

    (c) $3\langle 1, -1 \rangle - 2\langle 4, 6 \rangle = \langle 3, -3 \rangle - \langle 8, 12 \rangle = \langle -5, -15 \rangle$

    (d) $2(\hat{\imath} + \hat{\jmath}) + 5(2\hat{\imath} - 3\hat{\jmath}) = 12\hat{\imath} - 13\hat{\jmath}$ or $\langle 12, -13 \rangle$.

2. Find a unit vector in the direction of $\langle -3, 2 \rangle$.

    Since multiplying by a constant does not change the direction of the vector, we need to find a constant by which we can multiply to get a vector of length 1. This constant will be $1/$(length of vector). To get the length of the vector, we need:

    $$\|\langle -3, 2 \rangle\| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$$

    Then, the unit vector that we need is

    $$\frac{1}{\sqrt{13}}\langle -3, 2 \rangle = \left\langle -\frac{3\sqrt{13}}{13}, \frac{2\sqrt{13}}{13} \right\rangle$$

3. Find a unit vector in the direction of $4\hat{\imath} + 2\hat{\jmath}$.

    $$\|4\hat{\imath} + 2\hat{\jmath}\| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5},$$

    so the unit vector that we want is

    $$\frac{4\hat{\imath} + 2\hat{\jmath}}{2\sqrt{5}} = \frac{2\sqrt{5}}{5}\hat{\imath} + \frac{\sqrt{5}}{5}\hat{\jmath}$$

3   Finding Angles of Vectors

If $\mathbf{u} = \langle u_1, u_2 \rangle$ and θ is the angle that this vector makes with the positive x-axis, then $\tan \theta = \frac{u_2}{u_1}$. Note that you need to pay attention to the quadrant that the angle is in.

Examples
Find the direction angle of each of the following vectors:

1. $\mathbf{u} = \hat{\imath} + \sqrt{3}\hat{\jmath}$

    We need an angle θ such that $\tan \theta = \frac{\sqrt{3}}{1} = \sqrt{3}$, so $\theta = 60° = \frac{\pi}{3}$. Since the vector is in Quadrant I, we can use this angle. We could also use any coterminal angle, unless the directions for the question are more specific.

2. $\mathbf{u} = \langle -4, -4 \rangle$

    We need to find an angle θ such that $\tan \theta = \frac{-4}{-4} = 1$. However, since the vector is in Quadrant III, we need the angle to be in Quadrant III as well, so $\theta = 225° = \frac{5\pi}{4}$