Section 6.2, Law of Cosines

Homework: 6.2 #1, 3, 9, 31, 33, 37, 39

For oblique triangles, we know that $a^2 + b^2 \neq c^2$, but in this section, we will practice with the generalizations of that statement:

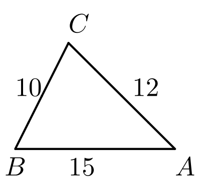
1 Law of Cosines

The **Law of Cosines** says that

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

This will help us when the Law of Sines doesn't work well (For example, when we are given the lengths of all 3 sides of the triangle, or when we are given the lengths of two sides as well as the measure between them). Note that these will give us at most one solution for each angle (unlike the Law of Sines, which can give two), since the sign ($\pm$) of value of cosine will determine whether the angle is in Quadrant I or II, which are the only angles allowed for triangles.

Examples
Solve each of the following triangles:



1.

We have the lengths of all 3 sides, $a = 10$, $b = 12$, and $c = 15$. To find the angle A, we can use

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ 10^2 &= 12^2 + 15^2 - 2\cdot 12\cdot 15\cdot\cos A \\ 100 &= 144 + 225 - 360\cos A \\ -269 &= -360\cos A \\ \cos A &= \frac{269}{360} \\ A &= \cos^{-1}\frac{269}{360} = 41.649^\circ \end{aligned}$$

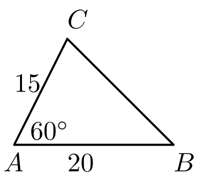
To find angle B, we can use:

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ 12^2 &= 10^2 + 15^2 - 2\cdot 10\cdot 15\cdot\cos B \\ 144 &= 100 + 225 - 300\cos B \\ -181 &= -300\cos B \\ \cos B &= \frac{181}{300} \\ B &= \cos^{-1}\frac{181}{300} = 52.891^\circ \end{aligned}$$

(We could have used the Law of Sines instead to find B, since we now have one angle.) To find angle C, we could use either the Law of Sines or the Law of Cosines. However, since we now know two of the three angles in the triangle, we can use that the three angles add up to $180°$, which gives

$$C = 180° - 41.649° - 52.891° = 85.460°$$

Note: The exact value of C rounds to $85.459°$, but due to rounding error, the subtraction method is off by 0.001. Normally, small rounding errors won't be a problem, but if you want a more exact answer, you can use the full decimal expansion shown in your calculator instead of a rounded version. You could also use the Law of Sines or Cosines to have a different approach.

2. 

From the diagram, we know that $A = 60°$, $b = 15$, and $c = 20$. To find the length of side a, we can use:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ a^2 &= 15^2 + 20^2 - 2 \cdot 15 \cdot 20 \cdot \cos 60° \\ a^2 &= 225 + 400 - 600 \cdot \frac{1}{2} \\ a^2 &= 325 \\ a &= \sqrt{325} = 5\sqrt{13} \end{aligned}$$

To find angle C, we can use

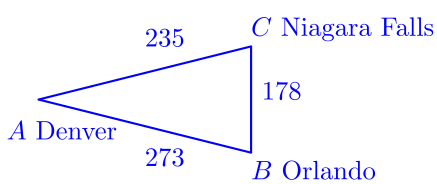
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ 20^2 &= (5\sqrt{13})^2 + 15^2 - 2 \cdot 5\sqrt{13} \cdot 15\cos C \\ 400 &= 325 + 225 - 150\sqrt{13}\cos C \\ -150 &= -150\sqrt{13}\cos C \\ \cos C &= \frac{1}{\sqrt{13}} = \frac{\sqrt{13}}{13} \\ C &= \cos^{-1}\frac{\sqrt{13}}{13} = 73.898° \end{aligned}$$

To find angle B, we can use that the sum of angles in a triangle is $180°$, so

$$B = 180° - 60° - 73.898° = 46.102°$$

3. (#37a) On a map, Orlando is 178 millimeters due south of Niagara Falls, Denver is 273 millimeters from Orlando, and Denver is 235 millimeters from Niagara Falls. Find the bearing of Denver from Orlando. (A "bearing" measures the acute angle that a path or line of sight makes with a fixed north-south line. It is defined on page 355 of the book. For example, a line that is 25° east of north is referred to as "N 25° E.")

   We can sketch a picture of this (there is also a diagram in the book):



With this diagram, $a = 178$, $b = 235$, and $c = 273$. We are only interested in angle B. To find it, we can use

$$\begin{aligned}
b^2 &= a^2 + c^2 - 2ac\cos B \\
235^2 &= 178^2 + 273^2 - 2 \cdot 178 \cdot 273 \cos B \\
-50988 &= -97188 \cos B \\
\cos B &= \frac{50988}{97188} \\
B &= \cos^{-1}\frac{50988}{97188} = 58.356°,
\end{aligned}$$

so the bearing is N 58.356° W.

Note: There is also part b of the question for you to try on your own in the homework.

4. We also set up the triangle needed for #39 in the homework.