Section 6.1, Law of Sines

Homework: 6.1 #1 (give exact answers), 3, 5, 7, 19, 20, 21, 35, 37

Note: Many of the homework problems will involve a calculator.

An **oblique triangle** is one with no right angles. There are two kinds of oblique triangles: acute and obtuse. A triangle is **acute** if all 3 angles are acute (less than 90°). A triangle is **obtuse** if one angle is obtuse (more than 90°).
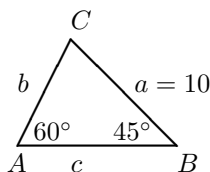
1   The Law of Sines

The **Law of Sines** says that for a triangle,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or}$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

(See page 430 of the book for the labeling of sides and angles of the triangle)

While this is designed to work for oblique triangles, works for right triangles. In this section, we cannot use $a^2 + b^2 = c^2$, since that only works for right triangles.

Examples

1. Solve the following triangle:

   The "quick" number to find is the measure of angle C, since we know that there are 180° in a triangle. Then, $C = 180° - 60° - 45° = 75°$.
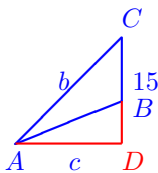
   To find b and c, we can use the law of sines:

   $$\frac{a}{\sin A} = \frac{b}{\sin B}$$
   $$\frac{10}{\sin 60°} = \frac{b}{\sin 45°}$$
   $$\frac{10}{\sqrt{3}/2} = \frac{b}{\sqrt{2}/2}$$
   $$b = \frac{20}{\sqrt{3}} \cdot \frac{\sqrt{2}}{2} = \frac{10\sqrt{2}}{\sqrt{3}} = \frac{10\sqrt{6}}{3}$$

   $$\frac{a}{\sin A} = \frac{c}{\sin C}$$
   $$\frac{10}{\sqrt{3}/2} = \frac{c}{\sin 75°}$$
   $$c = \frac{20}{\sqrt{3}} \cdot \frac{\sqrt{2}+\sqrt{6}}{4} = \frac{5(\sqrt{2}+\sqrt{6})}{\sqrt{3}} = \frac{5(\sqrt{6}+3\sqrt{2})}{3}$$

   where we used that $\sin 75° = \sin(30° + 45°)$ and used the sum formula from Section 5.4.

2. A 15-foot telephone pole that casts a 20-foot shadow directly down a slope is vertical. The elevation of the sun is 45°. Find the angle of elevation of the ground.

The red lines (and angle D) were added to the triangle described by the problem because the angle of elevation of the ground, θ, is the lower left angle in that triangle. Let A be the upper part of the bisected angle in the lower left corner of the diagram. Then, $a = 15$, $c = 20$, $C = 45$ (since it is part of a right triangle where one angle is $45°$), $A = 45° - \theta$

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$\frac{\sin(45° - \theta)}{15} = \frac{\sin 45°}{20}$$
$$\sin(45° - \theta) = \frac{15}{20} \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{8}$$
$$45° - \theta = \sin^{-1} \frac{3\sqrt{2}}{8}$$
$$\theta = 45° - \sin^{-1} \frac{3\sqrt{2}}{8} = 45° - 32.028° = 12.9722°$$

3. Solve the triangle with $A = 30°$, $a = 5$, and $b = 6$.

First, we can find angle B:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin 30°}{5} = \frac{\sin B}{6}$$
$$\sin B = \frac{6}{5} \cdot \frac{1}{2} = \frac{3}{5}$$
$$B = \sin^{-1} \frac{3}{5} = 36.870°$$

This isn't the only angle whose sine is $3/5$. $180° - 36.870° = 143.130°$ is a second solution. Since either angle added to $30°$ gives less than $180°$ (so there is "room" for a third positive angle in triangle), there are actually two solutions to the triangle:

$$B = 36.870°$$
$$C = 180° - 30° - 36.870° = 113.130°$$
$$c = \frac{a \sin C}{\sin A} = \frac{5 \cdot 0.9196}{1/2} = 9.196$$

and

$$B = 143.130°$$
$$C = 180° - 30° - 143.130° = 6.870°$$
$$c = \frac{a \sin C}{\sin A} = \frac{5 \cdot 0.1196}{1/2} = 1.196$$

There are many other solutions where there are 2 solutions.

4. Solve the triangle with $A = 120°$, $a = 10$, and $b = 11$.

Let's find B first. We know that:

$$\sin B = \frac{b \sin A}{a} = \frac{11\sqrt{3}/2}{10} = \frac{11\sqrt{3}}{20}$$

$$B = \sin^{-1} \frac{11\sqrt{3}}{20} = 72.294°$$

However, $A + B = 120° + 72.294° > 180°$, so there is no solution to this triangle. We could have gotten this without the Law of Sines by using that the side opposite the largest angle in a triangle is the longest side. Occasionally with this situation, we would get that the sine of an angle is greater than 1. (For example, if $b = 12$ instead, we would get $\sin B = 12\sqrt{3}/20 = 1.039$, which isn't possible.)

2 Area of Triangles

In general, the area of a triangle is $A = \frac{1}{2}$base $\cdot$ height. However, we aren't always given the height of the triangle. In this case, there is a formula:

$$\text{Area} = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B$$

Note that each letter appears once in each formula. Also, the angle in the formula is between the two sides that are used.

Example
Find the area of the triangle with $A = 30°$, $b = 20$, and $c = 15$.

$$\text{Area} = \frac{1}{2} bc \sin A = \frac{1}{2} \cdot 20 \cdot 15 \cdot \frac{1}{2} = 75$$