Section 5.5, Multiple-Angle and Half-Angle Formulas

Homework: 5.5 #23, 25, 27, 45–53 odds

Now, we will consider double-angle and half-angle formulas. In other words, we will take information that we know about an angle to find values of trigonometric functions for either double or half of that angle.

1 Double-Angle Formulas

$$\begin{aligned}\sin 2u &= 2\sin u\cos u\\ \cos 2u &= \cos^2 u - \sin^2 u\\ &= 2\cos^2 u - 1\\ &= 1 - 2\sin^2 u\\ \tan 2u &= \frac{2\tan u}{1-\tan^2 u}\end{aligned}$$

Each primary formula comes from a Sum Formula in the previous section with $u = v$. The alternate versions of the formula for $\cos 2u$ come from using that $\sin^2 u + \cos^2 u = 1$.

Examples

1. Find $\sin 2x$ and $\cos 2x$ if $\sin x = \frac{5}{6}$ and $0 \leq x \leq \frac{\pi}{2}$.
   We can first use $\sin^2 x + \cos^2 x = 1$ to get that $\cos x = \frac{\sqrt{11}}{6}$, so

   $$\sin 2x = 2\sin x\cos x = 2\frac{5}{6}\cdot\frac{\sqrt{11}}{6} = \frac{5\sqrt{11}}{18}$$

   Also,

   $$\cos 2x = 1 - 2\sin^2 x = 1 - 2\cdot\left(\frac{5}{6}\right)^2 = -\frac{7}{18}$$

   Note that we only needed the value of one trigonometric function for the $\cos 2x$ formula. Also note that if we needed $\tan 2x$, we could just calculate $\frac{\sin 2x}{\cos 2x}$.

2. Find $\sin 3x$ if $\sin x = \frac{5}{6}$ and $0 \leq x \leq \frac{\pi}{2}$.
   The given information is the same as in the previous question, so we can use the answers from those:

   $$\begin{aligned}\sin 3x = \sin(2x + x) = \sin(2x)\cos x + \cos(2x)\sin x &= \frac{5\sqrt{11}}{18}\cdot\frac{\sqrt{11}}{6} + \left(-\frac{7}{18}\right)\cdot\frac{5}{6}\\ &= \frac{55}{108} - \frac{35}{108} = \frac{20}{108} = \frac{5}{27}\end{aligned}$$

2 Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$
$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$
$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of $\sin\frac{u}{2}$ and $\cos\frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

Example
Calculate $\sin\frac{3\pi}{8}$, $\cos\frac{3\pi}{8}$, and $\tan\frac{3\pi}{8}$.
We will use the information that we know about $\frac{3\pi}{4}$ to find information about $\frac{3\pi}{8} = \frac{3\pi/4}{2}$. Also note that $\frac{3\pi}{8}$ is in Quadrant I, so all of our trigonometric functions will be positive.

$$\begin{aligned}
\sin\frac{3\pi}{8} &= \pm\sqrt{\frac{1-\frac{-\sqrt{2}}{2}}{2}} \\
&= \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2} \\
\cos\frac{3\pi}{8} &= \pm\sqrt{\frac{1+\frac{-\sqrt{2}}{2}}{2}} \\
&= \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2} \\
\tan\frac{3\pi}{8} &= \frac{\sin u}{1+\cos u} = \frac{\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2-\sqrt{2}}\cdot\frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{2\sqrt{2}+2}{2} = \sqrt{2}+1
\end{aligned}$$

Note: If we were asked about the angle $11\pi/8$, we could use the angle $u = 11\pi/4$, which is coterminal with $3\pi/4$. However, $11\pi/8$ is in Quadrant III, so we would need to choose the values of sine and cosine to be negative.