Section 5.4, Sum and Difference Formulas

Homework: 5.4 #3, 5, 11, 13, 19, 21, 37, 45, 51, 55, 61

In Chapter 4, we looked at finding the trigonometric functions of many angles. In this section, we will be able to find trigonometric functions for sums and differences of angles. This will allows for angles not listed on our unit circles such as $\pi/12 = \pi/4 - \pi/6$.

The formulas that we will use are:

$$\begin{aligned}
\sin(u+v) &= \sin u \cos v + \cos u \sin v \\
\sin(u-v) &= \sin u \cos v - \cos u \sin v \\
\cos(u+v) &= \cos u \cos v - \sin u \sin v \\
\cos(u-v) &= \cos u \cos v + \sin u \sin v \\
\tan(u+v) &= \frac{\tan u + \tan v}{1 - \tan u \tan v} \\
\tan(u-v) &= \frac{\tan u - \tan v}{1 + \tan u \tan v}
\end{aligned}$$

There are proofs of these on page 424 of your book. These were derived by a Greek astronomer named Hipparchus, who was born in 160 B.C. He is considered the inventor of trigonometry.

Examples

1. Calculate each of the following:

   (a) $\sin \frac{\pi}{12}$

   We can use that $\pi/12 = \pi/4 - \pi/6$ to get that

$$\begin{aligned}
\sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
&= \frac{\sqrt{6} - \sqrt{2}}{4}
\end{aligned}$$

   (b) $\cos \frac{7\pi}{12}$

   Note that $7\pi/12 = \pi/3 + \pi/4$.

$$\begin{aligned}
\cos \frac{7\pi}{12} &= \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\
&= \cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4} \\
&= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2} - \sqrt{6}}{4}
\end{aligned}$$

   (c) $\tan \frac{5\pi}{12}$

   We can use that $5\pi/12 = 3\pi/4 - \pi/3$ (There are many other sums and differences of angles that will work.)

$$
\begin{aligned}
\tan\frac{5\pi}{12} &= \tan\left(\frac{3\pi}{4}-\frac{\pi}{3}\right) \\
&= \frac{\tan\frac{3\pi}{4}-\tan\frac{\pi}{3}}{1+\tan\frac{3\pi}{4}\tan\frac{\pi}{3}} \\
&= \frac{-1-\sqrt{3}}{1+(-1)\sqrt{3}} \\
&= \frac{-1-\sqrt{3}}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}} \\
&= \frac{-4-2\sqrt{3}}{-2} = 2+\sqrt{3}
\end{aligned}
$$

2. Find $\sin(u-v)$ if $\sin u = \frac{12}{13}$, $\cos v = -\frac{4}{5}$, and both u and v are in Quadrant II.

$$
\begin{aligned}
\sin(u-v) = \sin u\cos v - \cos u \sin v &= \frac{12}{13}\cdot\left(-\frac{4}{5}\right)-\left(-\frac{5}{13}\right)\cdot\frac{3}{5} \\
&= \frac{-48+15}{65} = -\frac{33}{65},
\end{aligned}
$$

where $-\frac{5}{13}$ and $\frac{3}{5}$ are from the Pythagorean identity, $\sin^2\theta+\cos^2\theta = 1$.

3. Confirm the identity $\sin(x+y)+\sin(x-y) = 2\sin x\cos y$

$$
\begin{aligned}
\sin(x+y)+\sin(x-y) &= \sin x\cos y+\cos x\sin y+\sin x\cos y-\cos x\sin y \\
&= 2\sin x\cos y
\end{aligned}
$$

4. Verify that $\cos(\frac{\pi}{2}+x) = -\sin x$

$$
\begin{aligned}
\cos\left(\frac{\pi}{2}+x\right) &= \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x \\
&= -\sin x
\end{aligned}
$$