Section 4.7, Inverse Trigonometric Functions

Homework: 4.7 #1–15 odds, 37–61 odds

Our goal for this section will be to solve equations like $\sin x = 1/2$. In other words, we will be asked to find the angle that gives us a given value for a trigonometric function (sine, cosine, and tangent). We will do this by using inverse functions. If you are having trouble, you may find it useful to review the values of the trigonometric functions for common angles.

1 Defining the Inverse Trigonometric Functions

We will first focus on the inverse of the sine function. There are two different notations for the function, which are: $\arcsin x = \sin^{-1} x$ (not to be confused with $(\sin x)^{-1} = 1/(\sin x)$). They are defined as:

$$y = \arcsin x = \sin^{-1} x \quad \text{if} \quad \sin y = x,$$

where $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$. (The graph will be sketched in class.)

Examples
Find each of the following:

1. $\arcsin\left(\frac{1}{2}\right) = \pi/6$
2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\pi/3$
3. $\sin^{-1} 0 = 0$
4. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \pi/4$
5. $\arcsin 3$ DNE because the x-value is larger than 1

Next, we will consider the inverse of the cosine function. It is defined by

$$y = \arccos x = \cos^{-1} x \quad \text{if} \quad \cos y = x,$$

where $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$. (This graph will also be sketched in class.)

Examples
Find each of the following:

1. $\cos^{-1} \frac{1}{2} = \pi/3$
2. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 3\pi/4$
3. $\arccos 1 = 0$
4. $\cos^{-1}(-1) = \pi$

Lastly, let us consider the inverse of the tangent function. It is given by

$$y = \arctan x = \tan^{-1} x \quad \text{if} \quad \tan y = x,$$

where $-\infty < x < \infty$ and $-\pi/2 < y < \pi/2$. (This graph will also be sketched in class.)

Examples
Find each of the following:

1. $\tan^{-1} 1 = \pi/4$
2. $\arctan(-\sqrt{3}) = -\pi/3$

These functions allow us to solve equations for the angle. For example, we can solve $\cos\theta = 4/x$ for θ in terms of x as $\theta = \cos^{-1} \frac{4}{x}$.

2 Properties of Inverse Trigonometric Functions

There are some properties that allow us to cancel trigonometric functions with inverse trigonometric functions:

1. If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then $\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin y) = y$.
2. If $-1 \le x \le 1$ and $0 \le y \le \pi$, then $\cos(\cos^{-1} x) = x$ and $\cos^{-1}(\cos y) = y$.
3. If x is any real number and $-\pi/2 < y < \pi/2$, then $\tan(\tan^{-1} x) = x$ and $\tan^{-1}(\tan y) = y$.

Examples

1. $\tan(\tan^{-1} 43) = 43$
2. $\sin(\sin^{-1} -1/12) = -1/12$
3. $\cos^{-1}(\cos \frac{3\pi}{2}) = \frac{\pi}{2}$ since $3\pi/2$ is not in the range of inverse cosine. However, $\cos \frac{3\pi}{2} = \cos \frac{\pi}{2}$ and $\pi/2$ is in its range.
4. $\sec(\cos^{-1} \frac{4}{5}) = \frac{5}{4}$
5. $\sin(\cos^{-1} \frac{3}{5}) = \frac{4}{5}$, which we get by sketching and solving a triangle.
6. $\csc(\tan^{-1} -\frac{5}{12}) = -\frac{13}{5}$, which we can also get by sketching a triangle.
7. (#91 from the book, please refer to the book for the description and the diagram.)
   (a) We know that $\sin\theta = \frac{5}{s}$, so $\theta = \sin^{-1} \frac{5}{s}$.
   (b) We can use our answer from the previous part to get that $\theta = \sin^{-1} \frac{5}{40} = \sin^{-1} \frac{1}{8}$ when $s = 40$ and that $\theta = \sin^{-1} \frac{5}{20} = \sin^{-1} \frac{1}{4}$ when $s = 20$.