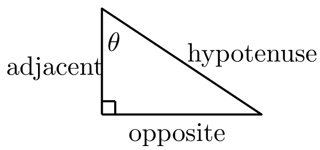
Section 4.3, Right Triangle Trigonometry

Homework: 4.3 #1–31 odds, 35, 37, 41

1 Another Approach for Calculating Trigonometric Functions

The techniques of this function work best when using acute angles, since we can draw any acute angle as part of a right triangle.

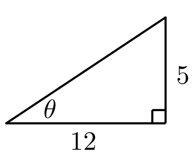


For the values of sine, cosine, and tangent, you can use "SOH-CAH-TOA."

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}$$
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Examples

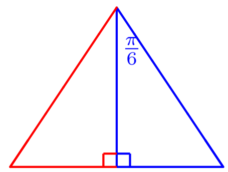
1. Find all 6 trigonometric functions for θ.



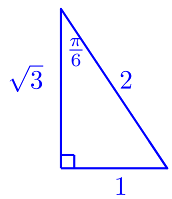
First, we need to calculate the length of the hypotenuse, which is $\sqrt{12^2 + 5^2} = \sqrt{169} = 13$ by the Pythagorean Theorem.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{13}{5}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{13}{12}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{12}{5}$$

2. Calculate all 6 trigonometric functions for $\pi/6$ (without using the unit circle). First, we will draw the triangle in blue, then include the "mirror image" of the triangle in red.



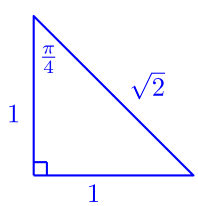
These two triangles combined give us an equilateral triangle (since all 3 angles are equal. Specifically, they all measure $\pi/6 = 60°$.). Let's assume that every side has length 2. Then, we can label the (smaller) triangle:



(The $\sqrt{3}$ is from using the Pythagorean Theorem). Then,

$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \csc\frac{\pi}{6} = 2$$
$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\frac{\pi}{6} = \sqrt{3}$$

3. Calculate all 6 trigonometric functions for $\pi/4$ (without using the unit circle).



This is an equilateral triangle, so the legs have the same length (here, we're assuming that it's 1). By the Pythagorean Theorem, the hypotenuse has length $\sqrt{2}$, so the 6 trigonometric functions are:

$$\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \csc\frac{\pi}{6} = \sqrt{2}$$
$$\cos\frac{\pi}{6} = \frac{\sqrt{2}}{2} \qquad \sec\frac{\pi}{6} = \sqrt{2}$$
$$\tan\frac{\pi}{6} = 1 \qquad \cot\frac{\pi}{6} = 1$$

Note: There is a table of values of sine, cosine, and tangent for "special" angles on page 303. Memorize $\theta = \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$.

Example
Let $\cot\theta = \sqrt{3}$ for $0 \le \theta \le \pi/2$. Find θ in radians.

By the table on page 303, we know that $\theta = \pi/6$.

2 More Trigonometric Properties and Identities

Cofunctions of complementary angles are equal. So, if θ is acute,

$$\sin(90^\circ - \theta) = \cos\theta \qquad \cos(90^\circ - \theta) = \sin\theta$$
$$\tan(90^\circ - \theta) = \cot\theta \qquad \cot(90^\circ - \theta) = \tan\theta$$
$$\sec(90^\circ - \theta) = \csc\theta \qquad \csc(90^\circ - \theta) = \sec\theta$$

90° can be replaced by $\pi/2$ if the angles are in radians.

Example
Let $\sin\theta = 2/3$.

1. Calculate $\cos(\frac{\pi}{2} - \theta)$.
   $\cos(\frac{\pi}{2} - \theta) = \sin\theta = 2/3$

2. Calculate all 6 trig functions for θ.

$$\sin\theta = \frac{2}{3} \qquad \csc\theta = \frac{3}{2}$$
$$\cos\theta = \frac{\sqrt{5}}{3} \qquad \sec\theta = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$
$$\tan\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \qquad \cot\theta = \frac{\sqrt{5}}{2}$$

   The $\sqrt{5}$ is from the Pythagorean Theorem.

The **Pythagorean Identities** are:

$\sin^2\theta + \cos^2\theta = 1$ Dividing both sides by $\cos^2\theta$, we get that:
$\tan^2\theta + 1 = \sec^2\theta$ Dividing both sides of the first line by $\sin^2\theta$, we get that:
$1 + \cot^2\theta = \csc^2\theta$

(Note: $\sin^2\theta = (\sin\theta)^2$.)

Examples
Show each of the following identities:

1. $(\sec\theta + 1)(\sec\theta - 1) = \tan^2\theta$
   When showing identities, it is normally easier to start with the more complicated side and simplifying it. Also note that there is more than one way to get to the correct answer.

$$\begin{aligned}(\sec\theta + 1)(\sec\theta - 1) &= \sec^2\theta - 1\\ &= \tan^2\theta\end{aligned}$$

2. $\dfrac{\tan\theta}{\sec\theta} = \sin\theta$

$$\begin{aligned}\frac{\tan\theta}{\sec\theta} &= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}\\ &= \sin\theta\end{aligned}$$

3. $\dfrac{\tan\beta + \cot\beta}{\tan\beta} = \csc^2\beta$ (#42)

$$\begin{aligned}\frac{\tan\beta + \cot\beta}{\tan\beta} &= 1 + \cot^2\beta\\ &= \csc^2\beta\end{aligned}$$