Section 4.2, Trigonometric Functions: The Unit Circle

Homework: 4.2 #1–41 odds

1 Trigonometric Functions

Instead of focusing on the angle, we will spend much of the semester focusing on the point (x, y) where the ray created by the angle crosses the unit circle.

First, note that $x^2 + y^2 = 1$ by the Pythagorean Theorem. (We'll discuss this in more detail later.)

Let θ be a real number and let (x, y) be the point on the unit circle corresponding to θ. Then, the six trigonometric functions (sine, cosine, tangent, cosecant, secant, and cotangent) are

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y} = \frac{1}{\sin\theta}, \quad y \neq 0$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}, \quad x \neq 0$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}, \quad y \neq 0$$

Examples
Calculate all 6 trigonometric functions for each of the following angles.

1. $\theta = 3\pi/4$
   This corresponds to the point $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ on the unit circle, so

$$\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2} \qquad \csc\frac{3\pi}{4} = \frac{1}{\sqrt{2}/2} = \sqrt{2}$$

$$\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \qquad \sec\frac{3\pi}{4} = -\frac{1}{\sqrt{2}/2} = -\sqrt{2}$$

$$\tan\frac{3\pi}{4} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1 \qquad \cot\frac{3\pi}{4} = \frac{1}{-1} = -1$$

   (Be sure to rationalize fractions!)

2. $\theta = \pi/2$
   This angle corresponds to $(0, 1)$ on the unit circle, so

$$\sin\frac{\pi}{2} = 1 \qquad \csc\frac{\pi}{2} = \frac{1}{1} = 1$$

$$\cos\frac{\pi}{2} = 0 \qquad \sec\frac{\pi}{2} = \frac{1}{0} = \text{undefined}$$

$$\tan\frac{\pi}{2} = \frac{1}{0} = \text{undefined} \qquad \cot\frac{\pi}{2} = \frac{0}{1} = 0$$

3. $\theta = 7\pi/6$

This angle corresponds to $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, so

$$\sin\frac{7\pi}{6} = -\frac{1}{2}$$
$$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$
$$\tan\frac{7\pi}{6} = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$
$$\csc\frac{7\pi}{6} = \frac{1}{-1/2} = -2$$
$$\sec\frac{7\pi}{6} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$
$$\cot\frac{7\pi}{6} = \sqrt{3}$$

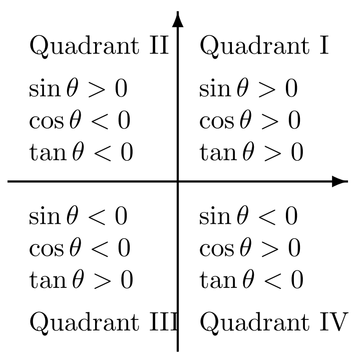
2 Some Notes on Trigonometric Functions

Note that

$$\begin{aligned} -1 &\le \sin\theta \le 1 \\ -1 &\le \cos\theta \le 1 \\ 1 &\le \left|\csc\theta\right| \\ 1 &\le \left|\sec\theta\right| \end{aligned}$$

There are no limitations of values for the values of $\tan\theta$ and $\cot\theta$.

The quadrant of the angle can help to determine the sign of the trigonometric functions:



All trigonometric functions are 2π-periodic. For example,

$$\begin{aligned} \sin\theta &= \sin(\theta \pm 2\pi) = \sin(\theta \pm 4\pi) = \cdots \\ \cos\theta &= \cos(\theta \pm 2\pi) = \cos(\theta \pm 4\pi) = \cdots \end{aligned}$$

This occurs because adding or subtracting 2π to an angle gives you a coterminal angle, so it can be represented by the same point on the unit circle.

Example
Evaluate each of the following:

1. $\sin\frac{11\pi}{4} = \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$
2. $\tan(-\frac{5\pi}{6}) = \tan\frac{7\pi}{6} = \frac{\sqrt{3}}{3}$

All of the trigonometric functions are either even or odd. cos and sec are even, so

$$\begin{aligned} \cos(-\theta) &= \cos\theta \\ \sec(-\theta) &= \sec\theta \end{aligned}$$

The rest of the functions are odd, so

$$\sin(-\theta) = -\sin\theta$$
$$\tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta$$
$$\cot(-\theta) = -\cot\theta$$

Examples

1. Given that $\sin(-\theta) = -3/4$, find
   (a) $\sin\theta = 3/4$
   (b) $\csc\theta = 4/3$
2. Given that $\cos\theta = 2/5$, find
   (a) $\cos(-\theta) = 2/5$
   (b) $\sec(-\theta) = 5/2$

Note: There are many other properties of trigonometric functions, but we have all semester to cover them!

Another Note: The book has some good diagrams on page 295.