Section 9.9, The Taylor Approximation to a Function

Homework: 9.9 #1–17 odds, 29–41 odds

There are many math problems that occur in applications that we cannot solve exactly, such as $\int_0^x e^{-t^2}\,dt$. If a solution is needed, we need to approximate them.

1 Taylor Polynomials

The **Taylor Polynomial** of order n based at a is given by

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The order is the last derivative that we take. Note that this is not necessarily the degree of the polynomial. For example, if $f^{(n)}(a) = 0$, it will be of degree at most $n-1$.

The **Maclaurin Polynomial** comes from using $a = 0$.

Example
For $f(x) = e^{-3x}$, find the Maclaurin polynomial of order 4 and use it to approximate $f(0.12)$.

For the Maclaurin polynomial, we first need derivatives evaluated at $x = 0$:

$$\begin{aligned} f'(x) &= -3e^{-3x} & f'(0) &= -3 \\ f''(x) &= 9e^{-3x} & f''(0) &= 9 \\ f'''(x) &= -27e^{-3x} & f'''(0) &= -27 \\ f^{(4)}(x) &= 81e^{-3x} & f^{(4)}(0) &= 81 \end{aligned}$$

Then, the Maclaurin polynomial of order 4 is

$$\begin{aligned} P_4(x) &= 1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \frac{81}{4!}x^4 \\ &= 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4 \end{aligned}$$

Then,

$$f(0.12) \approx P_4(0.12) = 1 - 3 \cdot 0.12 + \frac{9}{2} \cdot 0.12^2 - \frac{9}{2} \cdot 0.12^3 + \frac{27}{8} \cdot 0.12^4 = 0.69772384$$

Note: The actual answer is 0.6976763261.

2 Errors

Recall that the remainder (or error) caused by approximating a function by the n^{th} order Taylor polynomial is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

where $c \in (x, a)$. This is called the **Lagrange Error for Taylor Polynomials**.

Example
Find the error in estimating $f(0.12)$ in the last example.

We know that

$$\begin{aligned}\left|R_5(x)\right| &= \left|\frac{f^{(5)}(c)}{5!}x^5\right| = \left|\frac{-243e^{-3c}}{5!}\cdot 0.12^5\right| \\ &\leq \frac{243}{120}\cdot 0.12^5 = 5.038848\cdot 10^{-5},\end{aligned}$$

where we used that $e^{-3x} \leq 1$ when $x \in (0, 0.12)$ and $x = 0.12$.

Note: The actual error is $4.7513929 \cdot 10^{-5}$, which is less that what we calculated. (This is what should happen.)

The Lagrange Error formula gives us an error bound or the method itself. However, there may be additional rounding errors along the way due to the computations. This means that there is a balance for the number of terms that we need. Having more terms will reduce errors from the method itself, but increases the error caused by the calculations.

For example, consider a finite sequence $\{a_i\}_{i=1}^{n}$ where $a_i \approx 0.001$. If we have a number $s \approx 1,000,000$ and want to calculate $s - a_1 - a_2 - a_3 - \cdots - a_n$, we have two logical options:

1. We can find $s - a_1$, then $(s - a_1) - a_2$, then $\big((s - a_1) - a_2\big) - a_3$, and so on. This method may cause us to lose accuracy due to errors quickly.
2. We can first find $a_1 + a_2 + \cdots + a_n$, then find $s - (a_1 + a_2 + \cdots + a_n)$. This allows the computer or calculator to "focus" on the small numbers, and we aren't likely to lose as much accuracy due to rounding. This is probably the better method in this case.

To obtain a good bound for $R_n(x)$, we can use the triangle inequality, which says that $|a \pm b| \leq |a| + |b|$.

Examples
Find good bounds for the maximum value of each of the following on the given interval:

1. $\left|\dfrac{4c}{c+4}\right|$ for $c \in [0, 1]$

   Since we want to maximize the function, we can maximize the numerator and minimize the denominator, so we get

   $$\left|\frac{4c}{c+4}\right| \leq \frac{4}{c+4} \leq \frac{4}{4} = 1$$

   Note: You can get a slightly better bound by taking derivatives. The true maximum is $4/5$.

2. $\left|\tan c + \sec c\right|$ for $c \in [0, \pi/4]$.

   $$\left|\tan c + \sec c\right| \leq \tan\frac{\pi}{4} + \sec\frac{\pi}{4} = 1 + \sqrt{2}$$