Section 9.8, Taylor and Maclaurin Series

Homework: 9.8 #1–27 odds, 33

If we can represent a function $f(x)$ as a power series in $(x-a)$, then

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \\ f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \cdots \\ f''(x) &= 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + 20c_5(x-a)^3 + 30c_6(x-a)^4 + \cdots \\ f'''(x) &= 6c_3 + 24c_4(x-a) + 60c_5(x-a)^2 + 120c_6(x-a)^3 + 210c_7(x-a)^4 + \cdots \end{aligned}$$

and so on. Then, for $x = a$,

$$f(a) = c_0 \qquad f'(a) = c_1 \qquad f''(a) = 2c_2 \qquad f'''(a) = 6c_3 \quad \text{so}$$

$$c_0 = f(a) \qquad c_1 = f'(a) \qquad c_2 = \frac{f''(a)}{2} \qquad c_3 = \frac{f'''(a)}{6}$$

Generalizing, we see that $c_n = \dfrac{f^{(n)}(a)}{n!}$, so each c_n is unique and depends on the function f.

Uniqueness Theorem
Suppose that $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$ for all x in some interval around a. Then, $c_n = \frac{f^{(n)}(a)}{n!}$.

The series $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$ is called a **Taylor Series**. If $a = 0$, it is called a **Maclaurin Series**

Occasionally, we will need a finite sum instead of an infinite one. In this case, there will be an error introduced. If $f^{(n+1)}(x)$ exists for all x in an open interval I containing a, then for all $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where R_n is the remainder (or error). Then, $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, where c is some value between x and a.

Taylor's Theorem says that if f is a function with all derivatives in the interval $(a-r, a+r)$, then $\lim_{n\to\infty} R_n(x) = 0$, where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ and $c \in (a-r, a+r)$.

The formula for the remainder R_n will normally not be used to find the exact error. However, we can use it to find the maximum error in a given interval. We will consider the error more in the next section.

Examples

1. Write the Maclaurin Series for $f(x) = e^x$.

   For the formula, we need derivatives for $f(x) = e^x$:

   $$f'(x) = f''(x) = f''(x) = \cdots = e^x$$

   Then, the Maclaurin Series (the Taylor Series with $a = 0$) is

   $$\begin{aligned} e^x &= e^0 + e^0 x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \frac{e^0}{4!}x^4 + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \end{aligned}$$

2. Find the Maclaurin Series for $f(x) = \cos x$ and prove that it represents $\cos x$ for all x.

We need to find derivatives of $f(x) = \cos x$, so

$$\begin{aligned} f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \\ &\vdots \end{aligned}$$

Therefore,

$$\begin{aligned} \cos x &= \cos 0 - \sin 0 x - \frac{\cos 0}{2!}x^2 + \frac{\sin 0}{3!}x^3 + \frac{\cos 0}{4!}x^4 - \frac{\sin 0}{5!}x^5 - \frac{\cos 0}{6!}x^6 + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

To show that this holds for all values of x, we can show that the radius of convergence is infinite. Let $x \in \mathbb{R}$. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, so the Maclaurin series is an alternating series with terms that converge to 0. This means that the series holds for all values of x.

3. Find the Maclaurin Series for $f(x) = (1+x)^p$, where $p \in \mathbb{R}$.

If p is an integer, we know that

$$(1+x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \binom{p}{4}x^4 + \cdots$$

from the Binomial Formula, where

$$\begin{aligned} \binom{p}{k} &= \frac{p!}{k!(p-k)!} = \frac{p(p-1)(p-2)\cdots(p-(k-1))(p-k)!}{k!(p-k)!} \\ &= \frac{p(p-1)(p-2)\cdots(p-(k-1))}{k!} \end{aligned}$$

The last line of this identity actually holds for all $p \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. Then, for all $p \in \mathbb{R}$ and $|x| < 1$,

$$(1+x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \binom{p}{4}x^4 + \cdots$$

Note that if $p \in \mathbb{Z}^+$, $\binom{p}{k} = 0$ for $k > p$, so the infinite series becomes a finite sum, which is exactly the binomial formula.

4. Write the Maclaurin Series for $f(x) = (1-x^2)^{2/3}$ through the fifth term.

We could find this by taking derivatives, but this will get complicated quite quickly (After the first derivative, we would need the product rule at each step, which will introduce an extra term in each step.). To avoid this, we can first find the Maclaurin Series for $g(x) = (1+x)^{2/3}$, then evaluate it at $-x^2$ instead of x ($f(x) = g(-x^2)$). Since this uses the last example with $p = 2/3$, we can evaluate:

$$\begin{aligned} \binom{2/3}{1} &= 2/3 \\ \binom{2/3}{2} &= \frac{\frac{2}{3}\cdot\frac{-1}{3}}{2!} = -\frac{1}{9} \\ \binom{2/3}{3} &= \frac{\frac{2}{3}\cdot\frac{-1}{3}\cdot\frac{-4}{3}}{3!} = \frac{4}{81} \\ \binom{2/3}{4} &= \frac{\frac{2}{3}\cdot\frac{-1}{3}\cdot\frac{-4}{3}\cdot\frac{-7}{3}}{4!} = -\frac{7}{243} \end{aligned}$$

So,

$$\begin{aligned}(1+x)^{2/3} &= 1+\frac{2x}{3}-\frac{x^2}{9}+\frac{4x^3}{81}-\frac{7x^4}{243}+\cdots \\ (1-x^2)^{2/3} &= 1+\frac{2(-x^2)}{3}-\frac{(-x^2)^2}{9}+\frac{4(-x^2)^3}{81}-\frac{7(-x^2)^4}{243}+\cdots \\ &= 1-\frac{2x^2}{3}-\frac{x^4}{9}-\frac{4x^6}{81}-\frac{7x^8}{243}+\cdots\end{aligned}$$

for $|x|<1$

5. Find the Taylor Series for $f(x)=\sin x$ in $(x-\pi/4)$.

   Instead of $a=0$, we will be using $a=\pi/4$. To start, we need derivatives:

$$\begin{aligned} f(x) &= \sin x & f(\pi/4) &= \sin\frac{\pi}{4}=\frac{\sqrt{2}}{2} \\ f'(x) &= \cos x & f'(\pi/4) &= \cos\frac{\pi}{4}=\frac{\sqrt{2}}{2} \\ f''(x) &= -\sin x & f''(\pi/4) &= -\sin\frac{\pi}{4}=-\frac{\sqrt{2}}{2} \\ f'''(x) &= -\cos x & f'''(\pi/4) &= -\cos\frac{\pi}{4}=-\frac{\sqrt{2}}{2} \\ f^{(4)}(x) &= \sin x & f^{(4)}(\pi/4) &= \sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}\end{aligned}$$

   So, the Taylor series is

$$\begin{aligned}\sin x &= \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{2\cdot 2!}\left(x-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{2\cdot 3!}\left(x-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{2\cdot 4!}\left(x-\frac{\pi}{4}\right)^4+\cdots \\ &= \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(x-\frac{\pi}{4}\right)^4+\cdots\end{aligned}$$

6. Using $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\cdots$, write the Maclaurin Series up to the x^5 term for $f(x)=\frac{1}{1-\sin x}$.

   First, we can find the Maclaurin Series for $1-\sin x$:

$$\begin{aligned}1-\sin x &= 1-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\cdots\right) \\ &= 1-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\frac{x^9}{9!}+\cdots\end{aligned}$$

   Then, we can carry out long division with $1/(1-\sin x)$ to get our final answer. (Done in class.)

Note: If you are unsure of an answer, you can use a graphing utility to compare the original function to the first few terms of the series expansion. If the graphs are close near the value for $x=a$ ($x=0$ for the Maclaurin Series), there is a good chance that your answer is correct. If the graphs are not close, there normally either a mistake, or not enough terms are being graphed. For example, the terms listed for the expansion of $f(x)=\cos x$ (through the x^6 term) only give a good approximation for $x\in(-\pi/2,\pi/2)$. Including through the x^{12} term gives a good approximation for $x\in(-3\pi/2,3\pi/2)$. Since giving more terms makes the graphs more accurate, we can be fairly confident that the established pattern is the correct expansion.