# Section 9.7, Operations on Power Series

Homework: 9.7 #1–31 odds

In this section, it will be helpful to think of a power series as a polynomial with infinitely many terms.

## 1 Derivatives and Integrals of Power Series

**Theorem A** Assume that $S(x)$ is the sum of a power series on an interval $I$, so

$$S(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

If $x$ is interior to $I$, then:

1. $S'(x) = \sum_{n=0}^{\infty} D_x(a_n x^n) = \sum_{n=0}^{\infty} n a_n x^{n-1}$.
2. $\int_0^x S(t)\,dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

In other words, we can differentiate and integrate a power series one term at a time, and the radius of convergence is the same for $S(x)$, $S'(x)$, and $\int_0^x S(t)\,dt$. (Note: The convergence set may be slightly different, but the only difference will be whether or not it includes the endpoints)

**Examples**

1. We know that, for $-1 < x < 1$, $1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ by our formula for a geometric series. Then,

   $$\int_0^x \frac{1}{1-t}\,dt = \sum_{n=0}^{\infty} \int_0^x t^n\,dt = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} \text{ using the series expansion}$$

   $$\text{and } \int_0^x \frac{1}{1-t}\,dt = -\ln|1-t|\Big|_0^x = -\ln|1-x| + \ln 1 = -\ln(1-x) \text{ since } -1 < x < 1$$

   This gives us the formula

   $$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

   Moving the negative sign from the left side to the right, and using $x = -t$, we get the formula

   $$\ln(1+t) = \sum_{n=0}^{\infty} \frac{-1}{n+1}(-t)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^{n+1} = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

2. Show that $S'(x) = S(x)$ for $S(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$. Then find a formula for the sum of the series.

   First, we need to find the convergence set for $S(x)$. It turns out that the convergence set is the entire real number line (this example is similar to one in the previous section.). Taking the derivative, we get

$$\begin{aligned} S'(x) &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= S(x) \end{aligned}$$

Since $S(x) = S'(x)$, and the only function that we know that equals its derivative is $e^x$, we now know that $S(x) = e^x$.

3. Find a power series for $f(x) = \frac{1}{4+3x} = \frac{1/4}{1+\frac{3}{4}x}$

   Using the formula for the sum of a geometric series with $r = -\frac{3}{4}x$ and $a = 1/4$, we get that

$$\frac{1/4}{1+\frac{3}{4}x} = \sum_{n=0}^{\infty} \frac{1}{4}\left(-\frac{3x}{4}\right)^n$$

4. Find a power series for $f(x) = \frac{x}{(1+x)^2}$

   If we find a power series for $\frac{1}{(1+x)^2}$, we can quickly multiply it through by $x$ later.

   $\frac{1}{(1+x)^2}$ is the derivative of $-\frac{1}{1+x}$, so if we find the power series for $-\frac{1}{1+x}$, we can take its derivative. From the geometric series formula with $r = -x$ and $a = -1$, we know that

$$-\frac{1}{1+x} = \sum_{n=0}^{\infty} -(-x)^n,$$

   so

$$\frac{1}{(1+x)^2} = D_x\left[-\frac{1}{1+x}\right] = \sum_{n=0}^{\infty} D_x\left[-(-x)^n\right] = \sum_{n=0}^{\infty} n(-x)^{n-1}$$

   Then, multiplying by $x$, we get:

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} n(-x)^{n-1}x = \sum_{n=0}^{\infty} (-1)^{n-1}nx^n$$

## 2 Algebraic Operations with Power Series

Now, we will consider addition, subtraction, multiplication, and division. Addition and subtraction are straightforward, since we need only combine like terms. We will focus on multiplication and division.

**Theorem B** Let $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$, with both of these series converging for at least $|x| < r$. Then, $f(x) + g(x)$, $f(x) - g(x)$, and $f(x) \cdot g(x)$ converge for $|x| < r$. If $b_0 \neq 0$, the corresponding result holds for division, but we can only guarantee its validity if $|x|$ is sufficiently small.

**Examples**

1. Find the power series for $f(x) = \sinh x$.

   We know that $\sinh x = \frac{e^x - e^{-x}}{2}$. The power series expansions for $e^x$ and $e^{-x}$ are

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \cdots \\ &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \end{aligned}$$

When we subtract these, notice the cancellation of every other term:

$$\begin{aligned}\frac{e^x - e^{-x}}{2} &= \frac{0 + 2x + 0 + \frac{2x^3}{3!} + 0 + \cdots}{2} \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\end{aligned}$$

2. Find the power series for $f(x) = \frac{\arctan x}{1+x^2+x^4}$. (Hint: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.)

   We used long division in class to find that

$$\frac{\arctan x}{1 + x^2 + x^4} = x - \frac{4}{3}x^3 + \frac{8}{15}x^5 + \frac{23}{35}x^7 + \cdots$$

3. Find the following sum: $1 + x^2 + x^4 + x^6 + x^8 + \cdots$

   This is a geometric series with $r = x^2$ and $a = 1$, so

$$1 + x^2 + x^4 + x^6 + x^8 + \cdots = \frac{1}{1 - x^2}$$

4. Find the following sum: $\cos x + \cos^2 x + \cos^3 x + \cdots$.

   This is a geometric series with $r = \cos x$ and $a = \cos x$, so

$$\cos x + \cos^2 x + \cos^3 x + \cdots = \frac{\cos x}{1 - \cos x}$$