Section 9.6, Power Series

Homework: 9.6 #1–31 odds

In the last few sections, we have looked at series of constants. Now, we will transition to look at a series of functions. Our primary goal for this section will be to find the values of x for which the series converges. In future sections, we will find the function to which the series converges.

1 Power Series in x

A **Power Series in** x has the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

The **convergence set** of a power series is the set of x-values where the power series converges.

The convergence set for a power series $\sum a_n x^n$ is always an interval of one of the following three type:

1. The single point $x = 0$.
2. An interval of the form $(-R, R)$, $[-R, R]$, $[-R, R)$, or $(-R, R]$.
3. The entire real line, $(-\infty, \infty)$.

We say that the **radius of convergence** is 0, R, or ∞, respectively.

Examples

1. Find the convergence set for $\sum_{n=0}^{\infty} ax^n$, where $a \neq 0$ is a real number.

   This is the formula for a geometric series in x, so this converges for $|x| < 1$, so the convergence set is $(-1, 1)$.

2. Find the convergence set for $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$.

   We can start with the absolute ratio test. For this, we need a formula for the n^{th} term, which is $\frac{x^{2n}}{(2n)!}$ (Remember, the first term is actually when $n = 0$.) Then,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{|x|^{2(n+1)}}{(2(n+1))!}}{\frac{|x|^{2n}}{(2n)!}} \\ &= \frac{x^2}{(2n+2)(2n+1)} \end{aligned}$$

   Since taking $n > x$ shows us that this ratio converges to zero as $n \to \infty$ for all values of x, the convergence set is $\mathbb{R}$.

3. Find the convergence set for $1 + x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \frac{x^4}{\sqrt{4}} + \cdots$.

   We can write the n^{th} term of this as $\frac{x^n}{\sqrt{n}}$. Using the absolute ratio test gives us

$$\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}/\sqrt{n+1}}{|x|^n/\sqrt{n}} = |x| \cdot \frac{\sqrt{n}}{\sqrt{n+1}}$$

   This ratio limits to a number less than 1 when x is in the interval $(-1, 1)$, so it converges for those x-values. When $|x| > 1$, the ratio limits to a number larger than 1, so it diverges when

$|x| > 1$. This test is inconclusive for $x = \pm 1$, so we need to consider those cases separately. When $x = 1$, by the p-series test, this series diverges (using $p = 1/2$). When $x = -1$, this is an alternating series with terms that decrease to zero, so the series converges. Therefore, the convergence set is $[-1, 1)$.

2 Power Series in $(x - a)$

The power series that we've been looking at so far in this section have been centered around 0. Now, we will consider ones that are centered around another value, a.

A **Power Series in** $(x - a)$ has the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

The convergence set for a power series in a has one of three forms:

1. A single point $x = a$.
2. An interval of the form $(a - R, a + R)$ (possibly including one or both endpoints).
3. The whole real line $(-\infty, \infty)$.

Example

Find the convergence set for $1 + \frac{x-3}{2} + \frac{(x-3)^2}{2^2} + \frac{(x-3)^3}{2^3} + \cdots$

The n^{th} term of the series is $\frac{(x-3)^n}{2^n}$. Using the absolute ratio test, we get the ratio

$$\frac{a_{n+1}}{a_n} = \frac{|x-3|^{n+1}/2^{n+1}}{|x-3|^n/2^n} = \frac{|x-3|}{2}$$

This ratio is less than 1 when $x \in (1, 5)$. When $x = 1$ or $x = 5$, every term in the series is ± 1, so the series diverges. Therefore, the interval of convergence is $(1, 5)$.