Section 9.5, Alternating Series, Absolute Convergence, and Conditional Convergence

Homework: 9.5 #1–33 odds

For several sections, we have looked at series where every term is positive. Now, we will look at series with some negative terms.

An **alternating series** is one where every term has an opposite sign. For example,

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

where each $a_i > 0$.

1 Alternating Series Test

Let $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$ be an alternating series with $a_n > a_{n+1} >$ for all n. If $\lim_{n\to\infty} a_n = 0$, then the series converges.

The error made by estimating the sum S by S_n is less than or equal to a_{n+1}: $E_n = |S - S_n| \leq a_{n+1}$.

A proof of this was shown in class (it is also on page 475 on the book).

Examples

1. Does the **alternating harmonic series**, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, converge or diverge?

   Since this is an alternating series with decreasing terms that converge to zero, the series converges.

2. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$ converge or diverge? What is the largest error made by approximating the series by S_7?

   Since this is also an alternating series with decreasing terms that converge to zero, the series also converges.

   The error made by approximating S by S_7 is at most the 8^{th} term in the series:

   $$a_8 = \left|\frac{(-1)^9 8}{64+1}\right| = \frac{8}{65}$$

2 Absolute Convergence Test

The **Absolute Convergence Test** says that if $\sum |u_n|$ converges, then $\sum u_n$ also converges.

Example
Does the series $3 + \frac{3}{2^2} - \frac{3}{3^2} + \frac{3}{4^2} + \frac{3}{5^2} - \frac{3}{6^2} + \cdots$ converge or diverge?

Let u_n be the n^{th} term in this series. Then,

$$\sum |u_n| = \sum \frac{3}{n^2},$$

which converges, so the original series converges as well.

3 Absolute Ratio Test

We say that a series $\sum u_n$ **converges absolutely** if $\sum |u_n|$ converges. By the Absolute Convergence Test, we see that absolute convergence implies convergence. This means that all of the tests we considered in the last sections for positive terms still imply convergence in the case where some terms are negative. We can reconsider the ratio test:

The **Absolute Ratio Test** says that if $\sum u_n$ is a series of nonzero terms and that

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \rho$$

1. If $\rho < 1$ the series converges absolutely.
2. If $\rho > 1$ the series diverges.
3. If $\rho = 1$ the test is inconclusive. This means that the series may diverge, it may converge absolutely, or it may converge but not converge absolutely.

Example
Does the series $\sum (-1)^{n+1} \frac{n^2}{e^n}$ converge absolutely?

We can carry out the absolute ratio test:

$$\begin{aligned}\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} &= \lim_{n\to\infty} \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{e^n}{e^{n+1}} \\ &= \frac{1}{e},\end{aligned}$$

so the series converges absolutely.

4 Conditional Convergence

We say that $\sum u_n$ is **conditionally convergent** if $\sum u_n$ converges, but $\sum |u_n|$ diverges.

Example
Classify the following series as absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1} + \sqrt{n}}$$

Since this is an alternating series whose terms decrease to zero, we know that the series converges. Now, we need only check absolute convergence. In other words, we need to check if the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ converges. But,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n+1}},$$

which diverges. So, by the ordinary comparison test, this series diverges. Therefore, the given series is conditionally convergent.

The **Rearrangement Theorem** says that the terms of an absolutely convergent series can be rearranged without affecting the convergence of the sum of the series. This is not true for conditionally convergent series.