Section 9.4, Positive Series: Other Tests

Homework: 9.4 #1–35 odds

1 Geometric and p-Series

We have already determined when both geometric and p-series converge:

A geometric series, $\sum_{n=1}^{\infty} ar^n$ with $a \neq 1$, converges if $|r| < 1$ and diverges if $|r| \geq 1$.

A p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

2 Ordinary Comparison Test

The **Ordinary Comparison Test** says that if there exists a natural number N $0 \leq a_n \leq b_n$ for all $n \geq N$,

1. If $\sum b_n$ converges, so does $\sum a_n$.
2. If $\sum a_n$ diverges, so does $\sum b_n$.

Example

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{4n-3}{3n^2-2n-4}$ converge or diverge?

Since $\frac{4n-3}{3n^2-2n-4} \geq \frac{1}{n}$ for large n and $\sum \frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty} \frac{4n-3}{3n^2-2n-4}$ also diverges. (Note that if the coefficients for the leading terms in both the numerator and denominator were switched, this would still diverge due to this test combined with the linearity of series.)

3 Limit Comparison Test

The **Limit Comparison Test** says that if $a_n \geq 0$, $b_n \geq 0$ and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$ for $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge together. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

The Limit Comparison Test gets used more often than the Ordinary Comparison Test.

Example

Does the following series converge or diverge?

$$\frac{1}{1^2+1} + \frac{2}{2^2+1} + \frac{3}{3^2+1} + \dots$$

We can rewrite this series as $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. Consider the Harmonic Series, $\sum \frac{1}{n}$, which we know diverges. Then,

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Then, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges since the Harmonic Series diverges.

4 Ratio Test

The **Ratio Test** says that if $\sum a_n$ is a series of positive terms and that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

1. If $\rho < 1$, the series converges.
2. If $\rho > 1$, the series diverges.
3. If $\rho = 1$, the test is inconclusive.

   This test works very well in series that involve factorials and exponentials.

   Examples
   Determine the convergence or divergence of each of the following series:

   (a) $\sum_{n=1}^{\infty} \frac{n^3}{2n!}$

   We can try the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)^3}{2(n+1)!}}{\frac{n^3}{2n!}} \\ &= \lim_{n\to\infty} \frac{(n+1)^3 2n!}{n^3 2(n+1)!} \\ &= \lim_{n\to\infty} \frac{(n+1)^3 n!}{n^3 n!(n+1)} = 0,\end{aligned}$$

   so this series converges.

   (b) $3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!}$

   We can rewrite this sum as $\sum_{n=1}^{\infty} \frac{3^n}{n!}$. Then,

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \\ &= \lim_{n\to\infty} \frac{3^{n+1} n!}{3^n n!(n+1)} \\ &= \lim_{n\to\infty} \frac{3}{(n+1)} = 0\end{aligned}$$

   so this series converges.

   (c) $\sum_{n=1}^{\infty} \frac{n!}{3+n}$

   Since $\lim_{n\to\infty} \frac{n!}{3+n} = \infty$, this series diverges.