Section 9.3, Positive Series: The Integral Test

Homework: 9.3 #1–33 odds

1 Bounded Sum Test

A series $\sum a_k$ of nonnegative terms converges if and only if is partial sums are bounded above.

Examples
Determine the convergence or divergence of each of $\sum_{k=1}^{\infty} \frac{|\sin k|}{(k+1)!}$.

Note that each term in the sum is positive. We can use that $|\sin k| \leq 1$ to simplify what we are checking to

$$\sum_{k=1}^{\infty} \frac{|\sin k|}{(k+1)!} \leq \sum_{k=1}^{\infty} \frac{1}{(k+1)!}$$

If the partial sums of the second series are bounded above, we will know that the sum we are asked about converges by the bounded sum test. The n^{th} partial sum is

$$\begin{aligned} S_n &= \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots + \frac{1}{(n+1)!} \\ &\leq \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} \leq 1 \end{aligned}$$

Since this upper bound is a geometric series with $r = 1/2$, we know that the original sum we were given converges to a number that is at most 1.

2 Integral Test

If f is a continuous, positive, nonincreasing function on the interval $[1, \infty)$, then the infinite series

$$\sum_{k=1} f(k)$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)\, dx$$

converges.

Examples
Does $\sum_{k=1}^{\infty} \frac{3k^3}{2+k^4}$ converge or diverge?

Since $\frac{3x^3}{2+x^4}$ is positive, continuous, and nonincreasing on $[1, \infty)$, we can check the convergence of the integral

$$\int_1^{\infty} \frac{3x^3}{2+x^4}\, dx = \frac{3}{4} \ln|2 + x^4| \Big|_1^{\infty}$$

which diverges, so the sum diverges.

3 p-series Test

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-series. It converges if $p > 1$ and diverges if $p \leq 1$. The proof of this statement follows from the integral test for $p \geq 0$. For $p < 0$, this follows from the n^{th}-term test.

Example
Does $\sum_{k=4}^{\infty} \frac{1}{k^3}$ converge or diverge?

This sum converges since $p = 3 > 1$.

4 Approximating Errors

Let f be a positive, continuous, nonincreasing function on $[1, \infty)$, and let a_n be a sequence such that $a_n = f(n)$. Then, the error, E_n that we get by approximating the sum S by the first n terms of the series is

$$E_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots < \int_n^{\infty} f(x)\,dx$$

Example
Approximate the error made by approximating the series $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$ by the sum of the first 9 terms.

First, note that the sum converges, since we can rewrite the exponent in the denominator as 3/2. Then,

$$\begin{aligned} E_9 &< \int_9^{\infty} x^{-3/2}\,dx \\ &= -2x^{-1/2}\Big|_9^{\infty} \\ &= \frac{2}{\sqrt{9}} = \frac{2}{3} \end{aligned}$$