Section 9.2, Infinite Series

Homework: 9.2 #1–33 odds

1 Series

Consider the following scenario: Assume that you have a field of length 1. You start on one side of the field (at $x = 0$), and step halfway across the field, to $x = \frac{1}{2}$. Then, you keep stepping from where you are to halfway to the other side of the field. Following this pattern, you'll be at $x = \frac{3}{4}$ after the second step, then $x = \frac{7}{8}$ after the third step, etc. The problem is that you'll never make it to the other side of the field in any set number of steps. However, what happens if we can take infinitely many steps?

More formally, let S_n represent the sum of the lengths of the first n steps that you've taken. Then,

$$
\begin{aligned}
S_1 &= \frac{1}{2} = 1 - \frac{1}{2} \\
S_2 &= \frac{1}{2} + \frac{1}{4} = 1 - \frac{1}{4} \\
S_2 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{8} \\
&\vdots \\
S_n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}
\end{aligned}
$$

Then,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1,$$

so if it were possible to take infinitely many steps, we would make it to the other side of the field.

An **infinite series** is the sum of an infinite sequence:

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{i=1}^{\infty} a_i, \text{ or } \sum a_i \text{ if there is no confusion about how many terms to sum.}$$

The n^{th} **partial sum**, S_n is given by

$$S_n = \sum_{i=1}^{n} a_i$$

Definition of Convergence We say that the sum $\sum_{i=1}^{\infty} a_i$ **converges** and has the sum S if the sequence of partial sums $\{S_n\}$ converges to S. If $\{S_n\}$ diverges, then we say that the series **diverges** and has no sum.

2 Geometric Series

A **geometric series** has the form $\sum_{j=1}^{\infty} ar^{j-1}$, where $a \neq 0$ and $r \neq 0$.

Examples

1. Show that a geometric sum converges for $|r| < 1$ and diverges when $|r| \geq 1$. Find a formula for the sum if $|r| < 1$

   First, if $r = 1$, then $S_n = \sum_{j=1}^{n} a = na$. Since $a \neq 0$, the sequence of partial sums becomes infinite as $n \to \infty$, so the sequence diverges when $r = 1$.

   If $r \neq 1$,

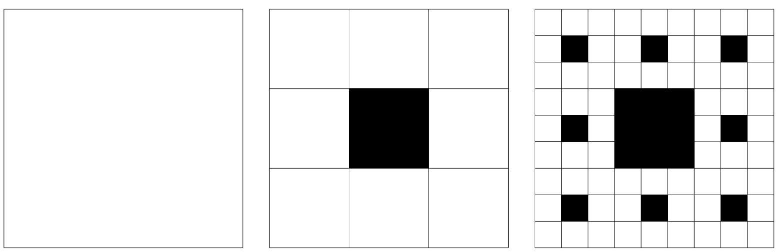
$$
\begin{aligned}
S_n - rS_n &= a - ar^n \\
S_n(1-r) &= a - ar^n \\
S_n &= \frac{a - ar^n}{1-r}
\end{aligned}
$$

   If $|r| < 1$, then $r^n \to 0$ as $r \to \infty$, so

$$\sum_{j=1}^{\infty} ar^j = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \frac{a}{1-r}$$

   so the sum converges. If $|r| > 1$ or $r = -1$, the sequence $\{r^n\}$ diverges, so the partial sums diverges.

2. If the pattern is continued indefinitely, what fraction of the original square will eventually be painted?



   If we call the empty box the 0^{th} box, then the amount shaded in the n^{th} box when $n \geq 1$ is

$$
\begin{aligned}
S_1 &= \frac{1}{9} \\
S_2 &= \frac{1}{9} + \frac{8}{9} \cdot \frac{1}{9} \\
S_3 &= \frac{1}{9} + \frac{8}{9} \cdot \frac{1}{9} + \left(\frac{8}{9}\right)^2 \cdot \frac{1}{9} \\
&\vdots \\
S_n &= \sum_{j=1}^{n} \frac{1}{9}\left(\frac{8}{9}\right)^{j-1},
\end{aligned}
$$

   so the limit of the partial sums is a geometric series. Then, the fraction painted is

$$\lim_{n\to\infty} S_n = \frac{1/9}{1 - 8/9} = 1$$

3 A Test for Divergence

The n^{th}**-Term test for Divergence** says that if $\sum_{n=1}^{\infty} a_n$ converges,then $\lim_{n\to\infty} a_n = 0$. This means that if $\lim_{n\to\infty} a_n \neq 0$ or does not exist, the series diverges.

Example

Does $\displaystyle\sum_{i=1}^{\infty} \frac{4i-7}{5i+4}$ converge or diverge?

Since $\lim_{i\to\infty} \frac{4i-7}{5i+4} = \frac{4}{5} \neq 0$, this series diverges.

Consider the **Harmonic Series**, $\sum_{n=1}^{\infty} \frac{1}{n}$. This series diverges, even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof of divergence: Consider the partial sums:

$$S_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$
$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{16}\right) \ldots + \frac{1}{n}$$
$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{n}$$

Since we can choose n sufficiently large to insert as many $\frac{1}{2}$'s as we wish, this series diverges. This means that finding that $\lim_{n\to\infty} a_n = 0$ is not enough to prove that $\sum_{n=1}^{\infty} a_n$ converges!

Example
Does $\sum_{i=1}^{\infty} \frac{3}{i(i+1)}$ converge or diverge?

Since $\lim_{i\to\infty} \frac{3}{i(i+1)} = 0$, this series might converge.

Since this isn't a geometric series, we need to try a different method. First, let's try to rewrite the fraction using partial fraction decomposition. Then, the fraction becomes:

$$\frac{3}{i(i+1)} = \frac{3}{i} - \frac{3}{i+1}$$

Then, writing the first few terms of the sum, we get:

$$\sum_{i=1}^{\infty} \frac{3}{i(i+1)} = \sum_{i=1}^{\infty} \left(\frac{3}{i} - \frac{3}{i+1}\right)$$
$$= \left(3 - \frac{3}{2}\right) + \left(\frac{3}{2} - \frac{3}{3}\right) + \left(\frac{3}{3} - \frac{3}{4}\right) + \ldots$$

Looking closely, we see that there is a lot of cancellation. It turns out that if we were to write infinitely many terms, everything except the 3 (the first part of the first term) would cancel! That means that this series converges to 3.

4 Linearity

If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ both converge, and c is a real number, then

$$\sum_{i=1}^{\infty} ca_i = c\sum_{i=1}^{\infty} a_i \text{ and}$$
$$\sum_{i=1}^{\infty} (a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$$

also converge.

Furthermore, if $\sum_{i=1}^{\infty} a_i$ diverges and $c \neq 0$, then $\sum_{i=1}^{\infty} ca_i$ also diverges.

We can also group the terms in a convergent series in any way, and this will give a new series that still converges to the same sum.

Examples
Determine whether the following series converge or diverge:

1. $\sum_{k=1}^{\infty} \left[4\left(\frac{4}{5}\right)^k - 2\left(\frac{1}{7}\right)^{k+1}\right]$

   If we look closely, we see that this is the difference of two geometric series. Since both of the geometric series converge, their difference does as well.

2. $\sum_{n=1}^{\infty} \frac{3}{n}$

Since this is a factor of 3 times the harmonic series (which diverges), this series also diverges.